Final Exam

PS 270 - Mathematical and Statistical Foundations

September 26, 2018

Linear Algebra & Calc

1. *2 points* Casstevens (1970) looked at legislative cohesion in the British House of Commons. Prime Minister David Lloyd George claimed on April 9, 1918 that the French Army was stronger on January 1, 1918 than on January 1, 1917 (a statement that generated considerable controversy). Subsequently the leader of the Liberal Party moved that a select committee be appointed to investigate claims by the military that George was incorrect. The resulting motion was defeated by the following vote: Liberal Party 98 yes, 71 no; Labour Party 9 yes, 15 no; Conservative Party 1 yes, 206 no; others 0 yes, 3 no. The difficulty in analyzing this vote results from the fact that 267 Members of Parliament (MPs) did not vote. So do we include them in the denominator when making claims about voting patterns? Casstevens says no because large numbers of abstentions mean that such indicators are misleading. Instead, he argued that we can evaluate party cohesion for the two large parties by comparing their respective vector norms for yes and no votes. The party with a vector of greater magnitude, he asserted, has more cohesiveness. Calculate the vector norms for each parties, and find which party exhibits greater cohesiveness.

2. *3 points* A mining company has two mines. One day's operation at mine #1produces ore that contains 20 metric tons of copper and 550 kilograms of silver. One day's operation at mine #2 produces ore that contains 30 metric tons of copper and 500 kilograms of silver. Let $v_1 = \left[\begin{smallmatrix} 20 \\ 550 \end{smallmatrix}\right]$ and $v_2 = \left[\begin{smallmatrix} 30 \\ 500 \end{smallmatrix}\right]$

   (a) What is the physical meaning of the vector $5v_1$?

   (b) Suppose the company operates mine #1 for x_1 days and mine #2 for x_2 days. Write an equation ($AX = B$) whose solution gives the number of days each mine should operate in order to produce 150 tons of copper and 2825 kilograms of silver.

   (c) What is the order of each of the matrices above?

3. *2 points* Compute the determinant $|A|$ of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 8 & 9 \end{bmatrix}$$

4. *2 points* Determine if the columns of the matrix B are linearly dependent.

$$B = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

5. *2 points* Show that AC + BC = (A+B)C.

$$A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

6. *3 points* In their formal study of models of group interaction, Bonacich and Bailey (1971) looked at linear and nonlinear systems of equations (their interest was in models that include factors such as free time, psychological compatibility, friendliness, and common interests). One of their conditions for a stable system was that the determinant of the matrix:

$R=1 \quad a=1$

$$B = \begin{bmatrix} -r & a & 0 \\ 0 & -r & a \\ 1 & 0 & -r \end{bmatrix}$$

must have a positive determinant for values of r and a. What is the arithmetic relationship that must exist for this to be true.

7. *2 points* Use the definition of the derivative to find the rate of change of the given function:

$$f(x) = x - 2x^2 + 3x^4$$

8. *2 points each* Find the derivatives of the given function, then find the instantaneous rate of change at the given point:

   (a) $f(x) = x^2 - x^3 + 2; f(1)$

   (b) $f(x) = \sqrt{x}(2x^2 - 5); f(4)$

   (c) $f(x) = \frac{4x^4 - 2x}{3x^2}; f(-1)$

9. *2 points* Use the second derivative test to classify critical points as local maxima or minima of the following function.

$$f(x) = x^3 - 9x^2 - 21x$$

10. *2 points each* Integrate the following indefinite integrals:

   (a) $\int(3x^2 + x - 4)dx$

   (b) $\int \frac{2x}{x^2+25}dx$

   (c) $\int xlnxdx$

11. *2 points each* Solve the definite integral.

   (a) $\int_1^3(2x + 5)dx$

   (b) $\int_{-2}^{-1}(\frac{2}{x^2})dx$

   (c) $\int_0^1 \frac{5e^{2x}}{(1+e^{2x})^{\frac{1}{3}}}dx$

12. *2 points* Find all first and second order partial derivatives of the following function:

$$f(x, y) = 3x^2y - 6xy^4 + (x + y)(x^3 - y^2)$$

13. *3 points* Consider

$$f(x, y) = (x - 1)^2 + y + 1$$

Find the critical point(s), and determine mix, max, or saddle point(s). Find ∇f. Calculate the Hessian matrix. Is f concave, convex, or neither?

14. *3 points* Candidate u selected campaign firm A to ensure he had a lead going into the last month of the election. Candidate v has less money so she selects campaign firm B, who cannot guarantee as many voters by the last month of the election. The firms model their hypothesized number of voters (in thousands) with the following equations, where m represents months from hiring.

$$A(m) = \frac{200m}{m + 5}$$

$$B(m) = 10(m - 3)^3$$

   (a) Assuming the final month of the campaign is 5 months away (m=5), does campaign firm A fulfill its promise?

   (b) Calculate A(5), B(5), A'(5) and B'(5)

   (c) Use the calculated values in part (b) to estimate the total votes received by each candidate on Election Day (m=6).

15. *3 points* In many parliamentary systems, the prime minister has the ability to dissolve parliament and call for new elections. Constitutionally, an election must be held within the next two years. It has been calculated that his party's populrity over the next 24 months can be modeled by

$$P(t) = \frac{8.4t}{t^2 + 49}; 0 < t < 24$$

When would you advise the election be held? Be sure to validate that this would be the best, and not the worst time to do so.

16. *3 points* In a newly incorporated city, voter registration (in thousands) for the first 8 months can be modeled by

$$V(t) = 30 + 12t^2 - t^3; 0 < t < 8$$

At what point is the amount of voters increasing at its maximum rate?

Probability

Note: Each question is worth four points except questions 9 and 12, where sub-sections are each worth four points.

Question 1. You own three coins. Each coin is a different color: one is silver, one is gold, and one is copper.

A) If you flip all three coins at once, what is the sample space over potential outcomes?

B) What outcomes make up event A in which you observe exactly two heads?

Question 2. You roll two six-sided dice simultaneously. The events A, B, and C are defined such that A = "both results are even"; B = "the sum of both results is nine or greater"; and C = "both of the results are five or greater."

A) Which events, if any, are mutually exclusive?

B) Which events, if any, are subsets of other events?

Question 3. $A = \{(x, y) : 0 < x < 5; 0 < y < 5\}$ and $B = \{(x, y) : 2 < x < 6; 2 < y < 6\}$. Given sets A and B, sketch the regions in the xy-plane the correspond to:

A) $A \cup B$

B) $A \cap B$

Question 4. An urn contains 36 chips, numbered 1-36. One chip is drawn at random. Let A be the event that the number on the chip is even. Let B be the event that the number of the chip is divisible by three.

A) Find $P(A \cup B)$

B) Find $P(A \cap B)$

Question 5. Suppose that two cards are drawn in order from a standard 52-card poker deck. In how many ways can one card be a heart and one card be a King?

Question 6. The La Jolla Lampreys are a local rugby club. This week they have two games: one against the Encinitas Echidnas and one against the San Marcos Muskox. The Lampreys have a 30% chance of winning against the Echidnas, but only a 20% chance of winning against the Muskox. If the Lampreys have a 60% chance of losing both games and a 10% chance of winning both games, what is their chance of winning exactly once?

Question 7. Let A and B be two events defined on a sample space Ω such that:

$$P(A \cap B^c) = 0.2$$

$$P(A^c \cap B) = 0.3$$

$$P((A \cup B)^c) = 0.1$$

Find the probability that at least one of the two events occurs given that at most one of the events occurs. Use set identities.

Question 8. You are in charge of selecting partners for the U.S. Olympic figure skating team. You are allowed to send four pairs of skaters to Olympics, with each pair consisting of one male and one female skater. You must choose these pairs from a set of 10 females and 10 males.

A) How many distinct pairs of skaters can you form?

B) How many different ways can you select *four* pairs of skaters?

C) The International Olympic Committee decides to eliminate rules that reify sexual binaries. Now you can create partnerships in which any two skaters can be paired together, regardless of sex or gender. In addition, the IOC decides to cut the number of entries in half. Under the new rules, how many different ways could you select *two* pairs of skaters from the 20 athletes?

Question 9. Urn I contains six red chips and ten white chips. Urn II contains three red chips and three white chips. Your friend chooses one of the two urns at random and draws a chip from that urn. (Each subsection is worth three points.)

A) If the chip is red, what was the probability that your friend sampled from Urn II?

B) You find a third urn containing five red chips and three white chips. Your friend selects a chip at random from one of the *three* urns. Given that the chip is red, what was the probability that your friend sampled from Urn III?

Question 10. On average, Stata crashes once per every 200 lines of code. To complete your seminar paper, you must execute 900 lines of code. What is the probability that more than 2 crashes will occur while you write your seminar paper?

Question 11. An urn contains five chips numbered 1 to 5. Two chips are drawn simultaneously.

A) Let X be the larger of the two numbers drawn. Write a probability mass function that describes X.

B) Let V be the sum of the two numbers drawn. Write a probability mass function that describes V.

C) Write the cumulative probability functions that describe X and V.

Question 12. An urn contains six chips numbered 2, 4, 6, 8, 10, and 12. Two chips are drawn without replacement. Let the random variable X denote the larger of the two chips. Each sub-question is worth three points.

A) Find E(X).

B) Find V(X)

Question 13. The leaders of Argentina are trying to decide whether to start a war with Bolivia. They know that the Bolivian military is either strong or weak. If Bolivia is weak, Argentina would win a war with 100% probability and receive a payoff of 5. If Bolivia is strong, Argentina would lose with certainty and receive a payoff of -5. If Argentina decides not to start a war, it will receive a payoff of 0.

A) If Argentina believes that Bolivia is strong with probability = .25 and not strong with complementary probability, what is Argentina's expected value of war?

B) Argentina observes Bolivia deploy a large number of troops to the border. This causes the Argentinian leadership to reassess their beliefs about Bolivia's strength. After much debate, they conclude that a strong Bolivia would be able to make such a move 60% of the time, but a weak Bolivia would be able to make the same move only $\frac{1}{3}$ of the time. After observing Bolivia's actions, should Argentina initiate war? What is their expected payoff?

Question 14. Given the cdf: $F(x) = x^{\frac{1}{2}}$

A) If the lower bound is 0, find the upper bound.

B) What is the expected value of this function?

C) What is the variance of this function?

Hint: To find expected value, derivate the CDF with respect to X, then integrate with respect to X including our X operator.