Game Theory Problem Set 1

Matthew Draper

10/12/18

1a. X: $\{x, y, z\}$ Y:$\{1, 2, 3\}$

X x Y = $\{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3), (z, 1), (z, 2), (z, 3)\}$

1b. X: $\{a, b, c, d\}$

X x X = $\{(a, a)(a, b)(a, c)(a, d)(b, a)(b, b)(b, c)(b, d)(c, a)(c, b)(c, c)(c, d)(d, a)(d, b)(d, c)(d, d)\}$

4. X: $\{x, y, z\}$; Suppose person i has the following preferences:

x$\succsim_i$x, x$\succsim_i$y, y$\succsim_i$y, y$\succsim_i$z, z$\succsim_i$z, z$\succsim_i$x

Restate in terms of $\succ$ and $\sim$

x$\succ_i$y, y$\succ_i$z, x$\succ_i$y, z$\succ_i$x, x$\sim$x, y$\sim$y, z$\sim$z

Preferences will cycle

(See Figure 1)

5a. C($\succsim$,X) = $\{w, z\}$ (X = $\{v, w, x, y, z\}$)

Relation: w$\succsim$v, w$\succsim$x, w$\succsim$y, z$\succsim$v, z$\succsim$x, z$\succsim$y, w$\sim$z

$\succsim$ is incomplete, intransitive

(See Figure 2)

5b. C($\succsim$,X) = $\{x\}$ (X = $\{v, w, x, y, z\}$)

Relation: x$\succsim$v, x$\succsim$w, x$\succsim$y, x$\succsim$z

$\succsim$ is incomplete, intransitive

(See Figure 3)

5c. C($\succsim$,X) = $\{v, w, x, y, z\}$ (X = $\{v, w, x, y, z\}$)

Relation: v$\succsim$w, v$\succsim$x, v$\succsim$y, v$\succsim$z, w$\succsim$v, w$\succsim$x, w$\succsim$y, w$\succsim$z, x$\succsim$v, x$\succsim$w, x$\succsim$y, x$\succsim$z, y$\succsim$v, y$\succsim$w, y$\succsim$x, y$\succsim$z, z$\succsim$v, z$\succsim$w, z$\succsim$x, z$\succsim$y

$\succsim$ is complete, transitive

(See Figure 4)

5d. C($\succsim$,X) = $\{\varnothing\}(X =${v, w, x, y, z})

Relation: none

$\succsim$ is incomplete, intransitive

(See Figure 5)

7. Theorem. If $\succsim$ on X is transitive, then $\succ$ and $\sim$ are also transitive.
Proposition is of the form: p $\rightarrow q$
Initial strategies: assume p or assume $\neg$ q (contrapositive)

Suppose $\neg$ q

Givens: suppose $\succ$ and $\sim$ are not transitve on x
Goal: $\succsim$ on X is not transitive

Step 1: Suppose $\exists$ x, y, z on X
Step 2: Suppose x $\succ$ y, y $\succ$ z, z $\succ$ x (cycle)
Step 3: Then x $\succ$ x $\implies$ contradiction ($\perp$)
Step 4: By Step 3, $\succsim$ on X is not transitive (Goal)
Step 5: $\neg$ q, therefore $\neg$ p

11. Theorem. if $\exists$ u : X $\rightarrow \mathbb{R}$ that represents $\succsim$ then $\succsim$ is complete and transitive. Proposition is of the form: p $\rightarrow q$
Initial strategies - assume p or assume $\neg$ q (contrapositive)

Suppose $\neg$ q

Givens: $\succsim$ is incomplete $\vee \succsim$ is intransitive
Goal: $\neg \exists$ u : X $\to \mathbb{R}$ that represents $\succsim$

Case 1: $\succsim$ is incomplete
Step 1: If $\succsim$ is incomplete then for some x, y $\in$ X such that $\neg$ (x $\succsim$ y) and $\neg$ (y $\succsim$ x), C($\succsim$ X) will be empty (by definition of the choice set).

Case 2: $\succsim$ is intransitive
Step 1: If $\succsim$ is intransitive, then for some x, y, z $\in$ X, preferences will cycle and the choice set will be empty (by definition of transitivity).

Step 2: In Cases 1 and 2 above, the choice set is empty for some values of x, y, z.

Axiom: Functions require a 1-to-1 correspondence between inputs and outputs.

Step 3: u : X $\to \mathbb{R}$ is a function

Step 4: If the choice set is empty for any values of x, y, z, then it cannot be modeled by a function.

Step 5: $\neg$ q, therefore $\neg$ p

19. A: (100,130); O: (70,90); P: (120,145)
a. Set of possible actions:{($\varnothing$, $\varnothing$), ($\varnothing$, A), ($\varnothing$, O), ($\varnothing$, P), (A, O), (A, P), (P, O), (A, A), (O, O), (P, P)}

b.
$Action : Payoff : Cost : NetBenefit :$
A,A 260 200 60
O,O 180 140 40
P,P 290 240 50
A,O 220 170 50
A,P 275 220 55
P,O 235 190 45
A 130 100 30

O	90	70	20
P	145	120	25
∅	0	0	0

c. Utility will be maximized by planting two apple trees. This is because the net benefit of planting two apple trees is 60, which is the highest achievable net benefit.

d. The action set is the same, but the optimal action has changed. Utility will now be maximized (55) by planting a pear tree and an apple tree.

Action:	Payoff:	Cost:	Net Benefit:
A,A	195	200	-5
O,O	135	140	-5
P,P	217.5	240	22.5
A,O	220	170	50
A,P	275	220	55
P,O	235	190	45
A	130	100	30
O	90	70	20
P	145	120	25
∅	0	0	0

21a. Yes, payoff function v also represents the same preferences as function u, because these preferences are ordinal, and order is preserved.

b. No, because order is not preserved ($u(a) < u(b)$ but $w(a) = w(b)$).

c. Let function t be such that $t(a) = 0$, $t(b) = 100$, and $t(c) = 400$.

d. Yes, let function s be such that $s(a) = -10$, $s(b) = -9$, and $s(c) = -6$. (My initial inclination here was to treat the new functions as linear transformations of u, which would make it impossible to make s(a) negative, but if we're really talking about ordinal functions, then it shouldn't matter that no scalar can transform zero into a negative number).

22. P(nomination) = .5; P(elected | nomination) = .4
u : (elected | nomination) = W;
u : $\neg$ (elected | nomination) = L;
u : (no action) = H

a. (See Figure 6)

b. The expected utility of running for Senate is:
E(u(elected | nomination) = W
E(u($\neg$ elected | nomination) = L
E(u(nominated | run) = (.4)(W) + (.6)(L)
E(u($\neg$ nominated | run) = H
E(u(run) = .5(.4W + .6L) + .5(H)
E(u(run) = .2W + .3L + .5H

c. H must be lower than the expected utility of running, which is:
E(u(run) = 5(.4W+.6L) + .5(H)

A reflection on the problem-solving process:

This problem set was frustrating but rewarding, in the sense that I learned more from the process of doing it than I had learned from reading the textbook. Game theory seems to be the type of activity where practice is essential, and if time permits I'm going to try to do the rest of the problems in the class notes. Learning LaTeX was difficult, and I'm embarrased to admit that I actually spent just as long typesetting as I spent solving the problems. That's just the initial learning curve, though - while the problem sets will get harder, typesetting them shouldn't, so next time I expect it to be less onerous. It is a skill worth having.

a) Initially, I solved the problems on my own, between Thursday 10/4 and Monday 10/8. Then, on Thursday 10/11, I met with Bianca and Kevin for an hour and reviewed our respective strategies . I was dismayed to find that they had taken different approaches to the proofs, but after reviewing my own approach I've retained it in both cases. I still think it's correct, but if it isn't I want to know specifically why not. It was really helpful to hear about how they approached the problems, particularly the proof strategies. Later on Thursday, I also discussed the problem set with Adam, but at a much more general level.

b) I used Tadelis extensively throughout the problem-solving process, and I also consulted Fudenberg and Tirole, though I'm not yet at a level where I can get much out of their text (beyond the basics). In the typesetting process, I used Google extensively, and compiled my LaTeX code at latexbase.com.

c) I'm struggling a bit with proofs, not because I find proof structure or techniques particularly difficult to use, but because of the open-ended nature of the problems. I would very much like to review problems 7 and 11 in class, and it might also be helpful to review lotteries over lotteries and how to reduce them. Other than that, I'm pretty comfortable with the rest of the material, and I hope I've demonstrated that here.

10/9/18

Game Theory Assignment #1

Figure #1:

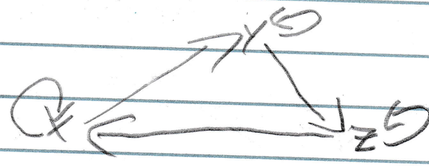
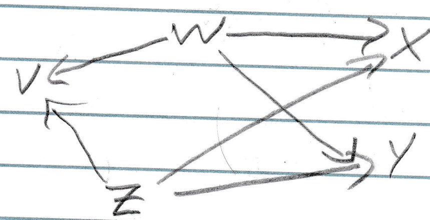
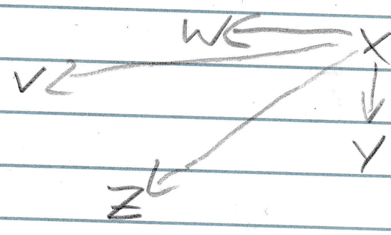


Figure #2:



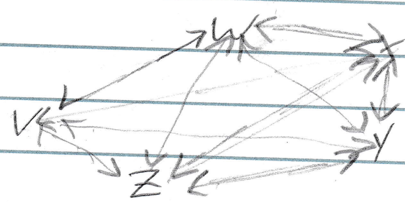
$\frac{W, Z}{V, X, Y}$

Figure #3:



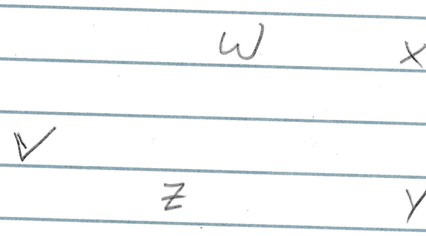
$\frac{X}{V, W, Y, Z}$

Figure #4:



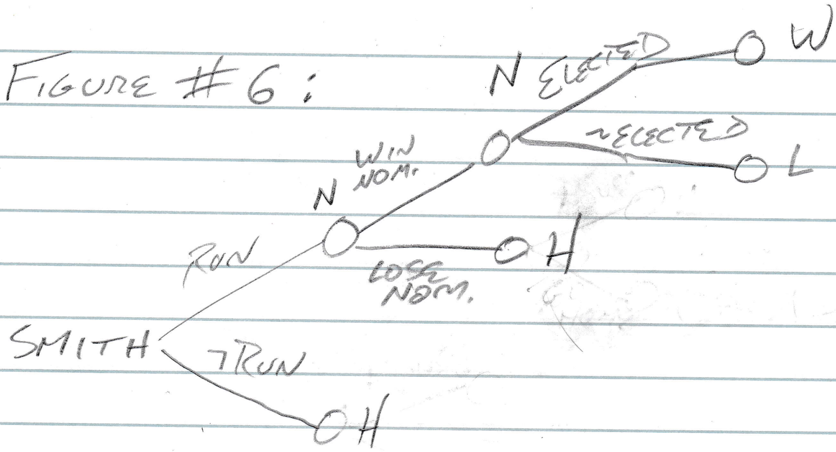
$\frac{V, W, X, Y, Z}{}$

Figure #5:

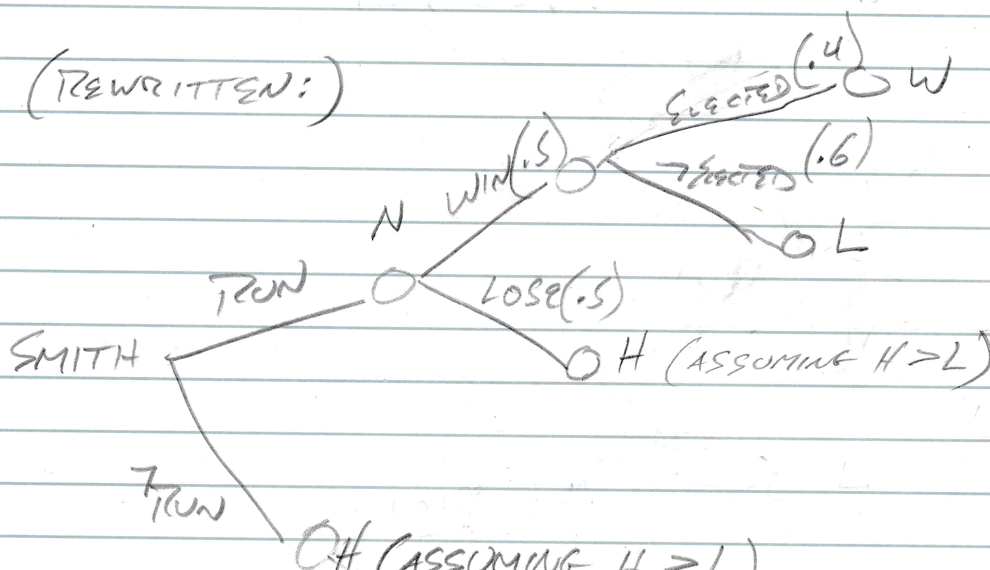


$\frac{}{V, W, X, Y, Z}$

FIGURE #6:



(REWRITTEN:)



Game Theory Problem Set 2

Matthew Draper

10/26/18

4. See Figure 1
5. See Figure 2

1. Theorem 4.1. If a game has a strictly dominant strategy equilibrium s^{D}, then s^{D} is the unique dominant strategy equilibrium.

Proof. Suppose s^{D} is not the unique dominant strategy equilibrium, but is still a strictly dominant strategy equilibrium. This means there must exist some other dominant strategy equilibrium, s^{D*} Then by the definition of equlibrium as a profile of mutually rational strategies (1.1), we can say that each player is maximizing expected utility at s^{D}. If each player is maximizing expected utility at s^{D}, then by the definition of strict dominance (4.2) they cannot also be maximizing utility at any s^{D*} Since they cannot be maximizing utility at any s^{D*}, s^{D*} cannot be a dominant strategy equilibrium.

7a. Prisoner's Dilemma: Best Response $\mathrm{BR}_1(\mathrm{S}_1) = \{\mathrm{D}\}$ if $(\mathrm{S}_2) = \{\mathrm{C}\}$; $\{\mathrm{D}\}$ if $(\mathrm{S}_2) = \{\mathrm{D}\}$; strategies are symmetrical for player 2.

7b. Pure Coordination: Best Response $\mathrm{BR}_1(\mathrm{S}_1) = \{\mathrm{L}\}$ if $(\mathrm{S}_2) = \{\mathrm{L}\}$; $\{\mathrm{R}\}$ if $(\mathrm{S}_2) = \{\mathrm{R}\}$; strategies are symmetrical for player 2.

7c. Coordination-conflict: Best Response $\mathrm{BR}_1(\mathrm{S}_1) = \{\mathrm{L}\}$ if $(\mathrm{S}_2) = \{\mathrm{L}\}$; $\{\mathrm{R}\}$ if $(\mathrm{S}_2) = \{\mathrm{R}\}$; strategies are symmetrical for player 2.

7d. "Hawk-Dove"/"Chicken": Best Response $\mathrm{BR}_1(\mathrm{S}_1) = \{\mathrm{S}\}$ if $(\mathrm{S}_2) = \{\mathrm{V}\}$; $\{\mathrm{V}\}$ if $(\mathrm{S}_2) = \{\mathrm{S}\}$; strategies are symmetrical for player 2.

7e. "Stag hunt": Best Response $\mathrm{BR}_1(\mathrm{S}_1) = \{\mathrm{S}\}$ if $(\mathrm{S}_2) = \{\mathrm{S}\}$; $\{\mathrm{H}\}$ if $(\mathrm{S}_2) = \{\mathrm{H}\}$; strategies are symmetrical for player 2.

7f. "Matching Pennies": Best Response $\mathrm{BR}_1(\mathrm{S}_1) = \{\mathrm{T}\}$ if $(\mathrm{S}_2) = \{\mathrm{H}\}$; $\{\mathrm{H}\}$ if $(\mathrm{S}_2) = \{\mathrm{T}\}$; strategies are symmetrical for player 2.

8. There is no strictly dominant strategy.
Player 2 will never play L.

Set of rationalizable strategies: $\mathrm{S}_1 = \{U, M, D\}; S_2 = \{C, R\}$

	L	C	R
U	(5*,1)	(1,4*)	(1,0)
M	(3,2)	(0,0)	(3*,5*)
D	(4,3)	(4*,4*)	(0,3)

We can eliminate column L
Now Player 1 will never play U. We can eliminate row U
After IESDS we are left with: $\mathrm{S}_2 = \{M, D\}; S_2 = \{C, R\}$

Suppose Player 2 plays a mixed strategy with a probability of p on C and probability of 1-p on R.

$\mathrm{u}_2(p, U) > u_2(L, U)$
$u_2(p, M) > u_2(L, M)$
$u_2(p, D) > u_2(L, D)$

$4p > 1; p > (1/4)$
$5(1-p) > 2; p < (3/5)$
$4p + 3(1-p) > 3; p > 0$

Player 2 should play a mixed strategy with p between (1/4) and (3/5)

Now we have: $\mathrm{S}_2 = \{p, M, D\}$

9. Preference orderings:

Prisoner's Dilemma:
Player 1: {(D,C), (C,C), (D,D), (C,D)}
Player 2: {(C,D), (C,C), (D,D), (D,C)}

Game #1:
Player 1: {(Y,X), (X,X), (X,Y), (Y,Y)}
Player 2: {(Y,X), (X,X), (X,Y), (Y,Y)}

Game #2:
Player 1: {(Y,X), (X,X), (Y,Y), (X,Y)}
Player 2: {(X,Y), (X,X), (Y,Y), (Y,X)}

Game #1 differs from the Prisoner's Dilemma in both action labels and preferences. Game #2 is equivalent to the Prisoner's Dilemma, differing only in the action labels.

19a. See Figure 3. This is a game of imperfect information because the decision between H and L and the decision between G and B are made simultaneously. Player 1 has two information sets, and Player 2 has only one.

19b. Player 1 will never play (A,L).

	G	B
A,H	(5*,3*)	(-4,2)
A,L	(2,-1)	(0,1*)
$\neg A, H$	(1,1*)	(1*,1*)
$\neg A, L$	(1,1*)	(1*,1*)

NE = {((A,H),G), ((¬ A,H),B), ((¬ A,L), B)

19c. The set of Nash equilibrium profiles is the intersection of the sets of rationalizable strategy profiles:

Set of rationalizable strategies: S = $S_1 x S_2$;
S_1 = {(A,H), (¬ A,H), (¬ A,L)}; $S_2 = \{G, B\}$

A reflection on the problem-solving process:

This problem set went more quickly than the last one, and I think I made fewer mistakes. I've been working through the non-assigned problems on my own, but it would be really helpful to see some worked examples (of rationalizable strategies, IESDS, best-response corresponcences, etc.), so as to pick up the correct notation and form.

a) Initially, I solved the problems on my own, between 10/19 and 10/22. I subsequently discussed strategies and approaches with Tom, Adam, Geoff, Kevin and Bianca.

b) I used Tadelis and Watson extensively throughout the problem-solving process. While typesetting, I consulted Google extensively and compiled my LaTeX code at latexbase.com.

c) I'm struggling with two things at the moment: notation and the connection between the techniques we're learning. Notation is just a matter of memorization, but it would be nice to review the connection between the methods and approaches in Notes 3 and come up with a rough algorithm for when to use each.

FIGURE 1

4. NORMAL FORM: PLAYERS ($I = \{1,2\}$)

STRATEGIES: $S_1 = (\varepsilon,\varepsilon',\varepsilon''),(\varepsilon,\varepsilon',o'')(\varepsilon o'\varepsilon'')(\varepsilon o' o'')$

$(o,\varepsilon',\varepsilon''),(o,\varepsilon',o'')(o o'\varepsilon'')(o,o',o'')$

$S_2 = (e,e',e'')(e,e',o'')(e,o',e'')(e,o',o'')$

$(o,e',e'')(o,e',o'')(o,o',e'')(o,o',o'')$

PAYOFF (UTILITY) FUNCTIONS - NO INFORMATION ON OUTCOMES

1\2	(e,e',e'')	(e,e',o'')	(e,o',e'')	(e,o',o'')	(o,e',e'')	(o,e',o'')	(o,o',e'')	(o,o',o'')
$(\varepsilon,\varepsilon',\varepsilon'')$	$P(Z_1)+(1-P)(Z_5)$							
$(\varepsilon,\varepsilon',o'')$								
$(\varepsilon,\varepsilon',o'')$								
$(\varepsilon,o',\varepsilon'')$								
(ε,o',o'')								
$(o,\varepsilon',\varepsilon'')$								
(o,ε',o'')								
(o,o',ε'')								
(o,o',o'')								

TWO GAMES: P AND 1-P

P { GAME P: $S_1 = \{\varepsilon, o\}$ $S_2 = \{(e,e')(e,o')(o,e')(o,o')\}$

$V_i(s_1,s_2)$; $V_1(\varepsilon,(e,e')) = V_2(\varepsilon,(e,o')) = Z_1$ → DO THIS FOR ALL

DO FOR 1-P

1\2	(e,e')	(e,o')	(o,e')	(o,o')
ε	Z_1	Z_1	Z_2	Z_2
o	Z_3	Z_4	Z_3	Z_4

1-P

1\2	e''	o''
$(\varepsilon',\varepsilon'')$	Z_5	Z_7
(ε',o'')	Z_5	Z_7
(o',ε'')	Z_6	Z_8
(o',o'')	Z_6	Z_8

FIGURE 2

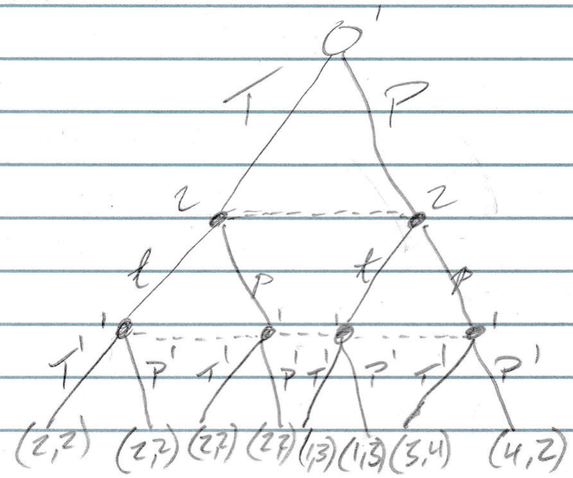
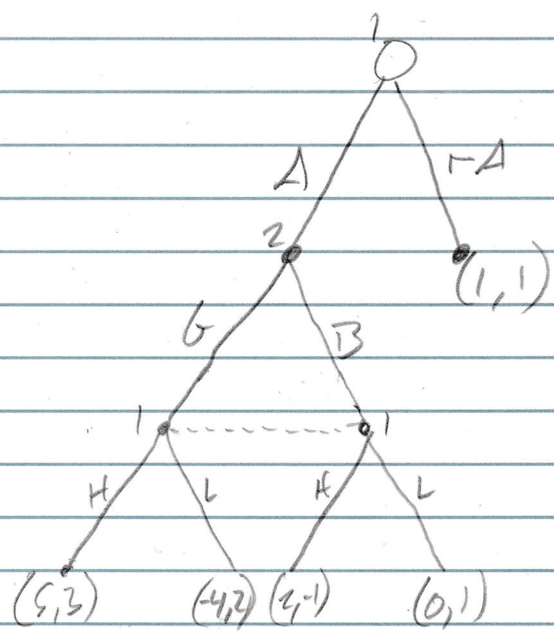


FIGURE 3



Game Theory Problem Set 3

Matthew Draper

11/9/18

21a) $I = \{1,2,3\}$; C = contribution, N = no contribution

$BR_1 (s_{-1}) = \{ \{C\}$ if $\{C, N\}$ or $\{N, C\}$; $\{N\}$ otherwise$\} \}$
$BR_2 (s_{-2}) = \{ \{C\}$ if $\{C, N\}$ or $\{N, C\}$; $\{N\}$ otherwise$\} \}$
$BR_3 (s_{-3}) = \{ \{C\}$ if $\{C, N\}$ or $\{N, C\}$; $\{N\}$ otherwise$\} \}$

21b) Four types of Nash equilibria – {no contribution, 1&2 contribute, 2&3 contribute, 1&3 contribute} : NE {1,2,3} = {N, N, N}, {N, C, C}, {C, N, C} and {C, C, N}

21c)
Let p be the probability of contribution by any player, and let (1-p) be the probability of non-contribution by any player. So:

Both players contribute: $p \times p = p^2$
One player contributes: $p(1-p) = p-p^2$
The other player contributes: $p(1-p) = p-p^2$
No player contributes: $(1-p)^2 = p^2 - 2p + 1$

$u_i (C, p) = u_i (N, p)$

$2(p^2) + 2(p-p^2) + 2(p-p^2) + -1(p^2 - 2p + 1) = 3p^2$

$p = 1/2 \pm (\sqrt{3})/6$ (0.78, 0.21)

21d) See Figure 1

21e)

1	**C**				**N**			
2\3	c''c'''	c''n'''	n''c'''	n''n'''	c''c'''	c''n'''	n''c'''	n''n'''
cc'	2,2,2	2*,2,2	2,2*,3*	**2*,2*,3***	**3*,2*,2***	0,-1,0	**3*,2*,2***	0,-1,0
cn'	2*,2,2	2*,2,2	**2*,2*,3***	2,2*,3*	0,0,-1	0,0*,0*	0,0,-1	0,0*,0*
nc'	2,3*,2*	**2*,3*,2***	-1,0,0	-1,0,0	**3*,2*,2***	0,-1,0	**3*,2*,2***	0,-1,0
nn'	**2*,3*,2***	**2*,3*,2***	-1,0,0	-1,0,0	0,0,-1	0,0*,0*	0,0,-1	**0*,0*,0***

The full set of 10 Nash equilibria appears (bold) in the table above. After checking each for single deviations, only 4 survive (underlined): { (C, nn', c''c'''), (C, cn', n''c'''), (N, cc', c''c'''), (N, nn', n''n''') }.

1) Whole game:

1\2	L	R
NE	4*,2	-1,3*
NW	0,5*	3*,1
SE	3,2*	**3*,2***
SW	3,2*	**3*,2***

NE = { (SE, R), (SW, R) } (bold)

Subgame:

1\2	L	R
E	4*,2	-1,3*
W	0,5*	3*,1

No pure-strategy Nash equilibria.

Check for mixed-strategy equilibria: suppose player 1 plays E with a probability of p and W with a probability of 1-p, and player 2 plays L with a probability of q and R with a probability of 1-q.

Then: u_1 (E, q) = u_1 (W, 1-q) and u_2 (L, p) = u_2 (R, 1-p)

4q – 1 + q = 5q-1 ; q = ½

2p + 5(1-p) = 3p + 1(1-p) ; p = 4/5

Mixed-strategy Nash equilibrium: { (4/5 E, 1/2 L) }

Candidate Subgame Perfect Equilibria: { (SE, R), (SW, R), (4/5 E, 1/2 L) }

In the first two equilibria, no deviation is possible for player 2 because they never get to play. For (SE, R) there are no profitable deviations for player 1. For (SW, R) player 1 is indifferent to a switch to N, but by proposition 4.1 this does not rule it out as subgame perfect. The mixed strategy appears to be subgame perfect. As a result, our set of subgame-perfect equilibria is: { (SE, R), (SW, R), (4/5 E, 1/2 L) }

3)

When candidates conduct their campaigns for office, they have two broad strategic options: they can either focus on issues (I) or on attacking their opponent (A). Most candidates would prefer to focus on the issues at stake in the campaign, but face the temptation to engage in personal attacks to improve their chances of victory. However, when *both* candidates engage in mudslinging its effects cancel out and the voters quickly tire of both of them (at least in this model). Now suppose that both candidates agree up front to stay focused on the issues. In a one-shot prisoner's dilemma, both candidates will defect and play A in spite of this mutual promise to play I, precluding the Pareto-optimal outcome.

Now imagine that the promise to focus on the issues is made in a public forum and that voters will swiftly punish a defector who goes negative after promising not to by refusing to vote for her in future races. We assume that voters are well-informed and able to distinguish between the candidate who goes negative and the candidate who does not. Behavior in the first-stage game will now be conditioned by expected payoffs in the second-stage "revenge" game.

Candidates will abide by their promise to focus on the issues and not go negative if the discounted penalties imposed by the voters are high enough to cancel out their gains from unilaterally playing A. This will depend on the values of the penalties (high enough negative numbers in the revenge game's lower-left (p2) and upper-right (p1) cells) and on each player's value of the discount factor (δ). A δ of 0 means that the players don't take the second stage into account at all (maximally impatient), and a δ of 1 means that the players value the second-stage payoffs as much as the first (maximally patient). Crucially, the second-stage game must also contain multiple Nash equilibria (as the revenge game does), otherwise threats to change strategy will not be credible. If these conditions are met, the candidates will be able to arrive at the Pareto-optimal outcome in the first-stage game.

This analysis reveals that candidates' perceptions of the sanctions voters will impose condition their behavior, and that if the potential (discounted) punishment is severe enough, politicians can be nudged towards campaigning on the issues rather than engaging in unproductive mudslinging.

9a) See Figure 2. The game has two proper subgames, assuming that the normal-form table indicates simultaneous moves (if the moves were sequential, it would have four).

a-bis) Imagine a prime minister calling an election in a parliamentary system. The prime minister must decide to call (C) or not call (N) the election. The game ends if the election is not called. If the election is called, then the prime minister must decide whether or not to accept preemptive offers of coalition or to run without any partners. At the same time the prime minister makes this decision, the leader of a small opposition party must decide whether to actively contest the election or to propose a coalition. In this scenario, the players are the prime minister and the leader of the opposition.

The players' strategy options are: S_{PM} = { Call, Coalition, Call, ¬Coalition, ¬Call, Coalition, ¬Call, ¬Coalition } ; S_{LO} = { Coalition, Contest }. Player PM will get a benefit of 0 if the election is contested and a benefit of 8 from an offer of coalition, while Player LO will get a benefit of 0 from contesting the election (since their party is small, they don't have a realistic chance of securing a majority whether or not they contest), and a benefit of 2 from entering into coalition. If they enter into coalition, both will receive a benefit of 6 if LO enters into coalition, and both will receive a benefit of 2 if the offer of coalition is rejected (imagine that this is interpreted as an expression of confidence in each party's electoral prospects.

b) S_A = {SU, SD, PU, PD} ; S_B = {L, R}

A\B	L	R
SU	3,3*	**3*,3***
SD	3,3*	**3*,3***
PU	8*,0	0,2*
PD	6,6*	2,2

c) Pure-strategy NE = {(SU, R), (SD, R)}

Mixed-strategy NE: suppose player A plays U with a probability of p and D with a probability of 1-p, and player B plays L with a probability of q and R with a probability of 1-q.

Then: u_A (U, q) = u_A (D, 1-q) and u_B (L, p) = u_B (R, 1-p)
8q = 4q+2 ; q = 1/2
6-6p = 2 ; p = 2/3
Mixed-strategy Nash equilibrium: { (1/2 U, 2/3 D) }

d)
Candidate Subgame Perfect Equilibria: { (SU, R), (SD, R), (1/2 U, 2/3 D) }
In the first two equilibria, no deviation is relevant for player B because they never get to play. For (SU, R) and (SD, R), player A has no profitable deviations because if player B is playing R, player A's maximum payoff is 2, and player A gets 3 by staying at S. The mixed strategy appears to be subgame perfect. As a result, our set of subgame-perfect equilibrium is: { (1/2 U, 2/3 D) }

e)
If we change A’s payoff in (PD, R) to 4 (from 2), then (SU, R) and (SD, R) will no longer be Nash equilibria because holding B’s strategy of R fixed, 4 is the highest payoff available to A and thus the payoff of 3 available in (SU, R) (SD, R) is no longer preferred.

12) a)
NE (t=0) = { N,N }
NE (t=1) = { (P,p), (N,n) }

1\2	p	n
P	2,2	0,5*
N	5*,0	3*,3*

1\2	p	N
P	6*,6*	1,0
N	0,1	2*,2*

{P,p} 1\2	p	N
P	**$2+6\delta$** **$2+6\delta$**	$2+1\delta$ $2+0\delta$
N	$2+0\delta$ $2+1\delta$	**$2+2\delta$** **$2+2\delta$**

{P,n} 1\2	p	N
P	**$0+6\delta$** **$5+6\delta$**	$0+1\delta$ $5+0\delta$
N	$0+0\delta$ $5+1\delta$	**$0+2\delta$** **$5+2\delta$**

(All NE in bold)

{N,p} 1\2	p	N
P	**$5+6\delta$** **$0+6\delta$**	$5+1\delta$ $0+0\delta$
N	$5+0\delta$ $0+1\delta$	**$5+2\delta$** **$0+2\delta$**

{N,n} 1\2	p	N
P	**$3+6\delta$** **$3+6\delta$**	$3+1\delta$ $3+0\delta$
N	$3+0\delta$ $3+1\delta$	**$3+2\delta$** **$3+2\delta$**

When $\delta = 0$, players are maximally impatient and don’t care about the second stage. This collapses the game to the first stage’s payoffs.

S_1 = { N, P,P,P,N }
S_2 = {n, p, n, p, n }

b)
$S_1 = \{ P,P,N,N,P \}$
$S_2 = \{ p,p,n,n,p \}$
(If P,p at the first stage, then 2 has a single profitable deviation to p, but 1 could counter by threatening to switch to N, since 1 has a higher payoff at N,n than at P,n).

c)
$2 + 6\delta \geq 3 + 2\delta$
$6\delta \geq 1 + 2\delta$
$4\delta \geq 1$
$\delta \geq 1/4$

d)
Consider the potential deviation (P,NPNN) (n, npnn)

$6\delta \geq 3+2\delta$
$\delta \geq 3/4$
Player 1 will have an incentive to deviate from (P,p) (N,n) to the proposed strategy if their value of $\delta \geq 3/4$.

$5+6\delta \geq 2+ 2\delta$
$\delta \geq -3/4$
Player 2 will never have an incentive to deviate from (P,p) (N,n) ($\delta \geq -3/4$)

FIGURE 1

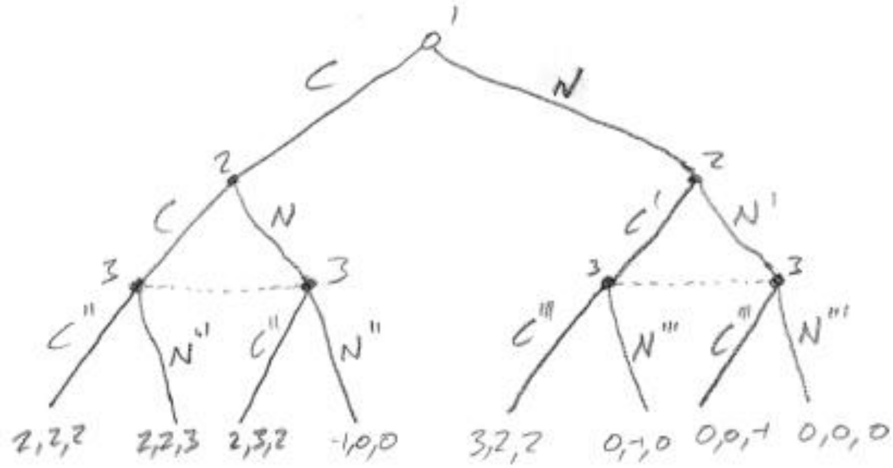
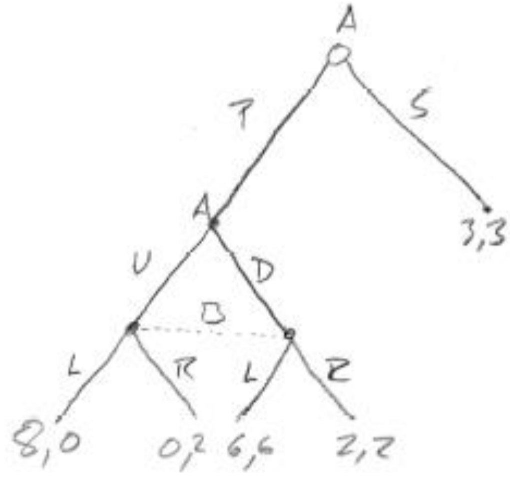


FIGURE 2



A reflection on the problem-solving process:

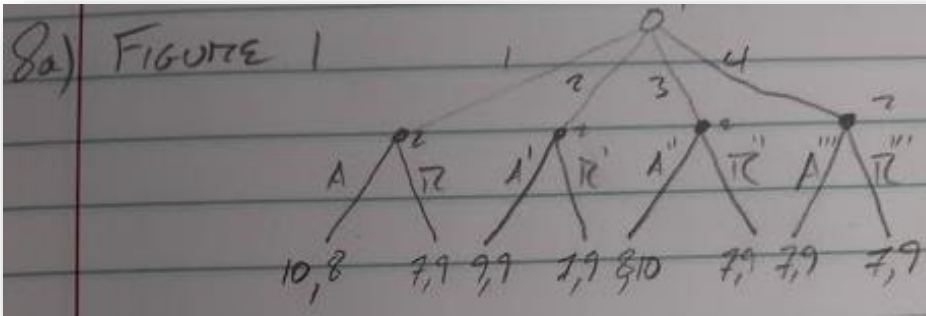
a) This problem set is dramatically more difficult than the last two, and I didn't have as much time to work on it because the midterm exam crowded out other priorities until 11/4. I started working on the problem set 11/5 – 11/6, and then I reviewed general approaches with Terry, Tom, Adam, Bianca, Kevin, and Patrick.
b) I used Tadelis and Watson throughout the problem-solving process.
c) I'm struggling with multi-stage games. I plan to use the long weekend to extensively review the notes, and I'll come in to office hours if there are concepts I'm still not following.

Game Theory Problem Set 4

Matthew Draper

11/26/18

8a) See Figure 1

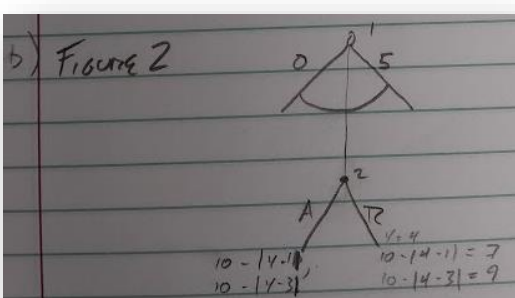


This game has a total of 18 Nash equilibria. They are in bold in the table below.

1\2	**1**	**2**	**3**	**4**
AA'A''A'''	10*,8	9*,9	**8*,10***	7*,9
AA'A''R'''	10*,8	9*,9	**8*,10***	7*,9
AA'R''A'''	10*,8	**9*,9***	7,9*	7*,9
AR'A''A'''	10*,8	7,9	**8*,10***	7*,9
AA'R''R'''	10*,8	**9*,9***	7,9*	**7*,9***
AR'R''A'''	10*,8	7,9*	7,9*	**7*,9***
AR'A''R'''	10*,8	7,9	**8*,10***	7*,9
AR'R''R'''	10*,8	7,9*	7,9*	**7*,9***
RA'A''A'''	7,9	9*,9	**8*,10***	7*,9
RA'A''R'''	7,9	9*,9	**8*,10***	7*,9
RA'R''A'''	7,9*	**9*,9***	7,9*	**7*,9***
RR'A''A'''	7,9	7,9	**8*,10***	7*,9
RA'R''R'''	7,9*	**9*,9***	7,9*	**7*,9***
RR'R''A'''	7,9*	7,9*	7,9*	**7*,9***
RR'A''R'''	7,9	7,9	**8*,10***	7*,9
RR'R''R'''	7,9*	7,9*	7,9*	**7*,9***

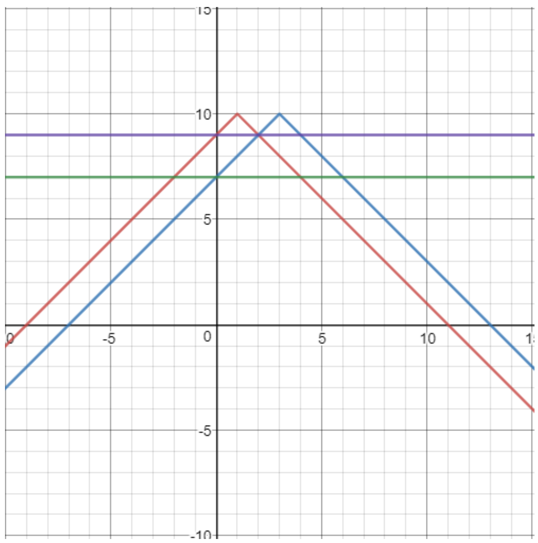
The equilibrium at (2,RA'A''A''') is subgame perfect because neither player has a single profitable deviation. Player 2 will not deviate because they can't do any better than 9 (assuming Player 1 plays 2). Player 1 will not deviate because 9 > 7 (the only alternative payoff available among all three options).

8b) See Figure 2



8c) We will examine this game by graphing the equations in Figure 2.

Red: Player 1’s utility function
Purple: Player 1’s reservation value (utility of the status quo)
Blue: Player 2’s utility function
Green: Player 2’s reservation value (utility of the status quo)



The intersection of the red and blue lines is the unique subgame perfect equilibrium for this version of the game (variables have been inverted for ease of graphing). Both players have an “acceptance space” constituted (for Player 1) by the red triangle above y = 9, and (for Player 2) by the blue triangle above y = 7. The only place those acceptance spaces intersect is at x = 2. This implies (in the original equations) a y value of 2 and a consequent payoff of 9 for both players. This subgame perfect equilibrium is similar to the one we characterized in part a). Player 1’s strategy will be to propose 2. Player 2’s strategy will be to accept any offer between 2 and 4 inclusive, and reject otherwise.

8d) There is a Nash equilibrium where player 2 will only accept 3 (and this is common knowledge). Then Player 1 would propose 3, because any other proposal would result in a lower payoff. This is a (non-subgame-perfect) Nash equilibrium because both players are playing best responses. There is an infinite number of similar n-s-p NEs – consider 2.999 and 3.001.

1a)

1\2	U	D
LL'	2(p)+2(1-p)	2(p)+2(1-p)
	0(p)+0(1-p)	0(p)+0(1-p)
LR'	2(p)+0(1-p)	2(p)+4(1-p)
	0(p)+0(1-p)	0(p)+2(1-p)
RL'	0(p)+2(1-p)	4(p)+2(1-p)
	4(p)+0(1-p)	0(p)+0(1-p)
RR'	0(p)+0(1-p)	4(p)+4(1-p)
	4(p)+0(1-p)	0(p)+2(1-p)

1b)

1\2	U	D
LL'	**2*,0***	2,0*
LR'	1,0	3,1*
RL'	1,2*	3,0
RR'	0,2*	4*,1

For p = 1/2, there is a Bayesian Nash equilibrium at (LL', U)

4a) Player 1 is the moderate type.

1\2	C_MC_V	A_MC_V	C_MA_V	A_MA_V
C_M	3	3-3p	3p	0
	p+2	2	3	3-p
A_M	2	2-p	p+1	1
	0	p	1-p	1

4b) Now we work out best responses:

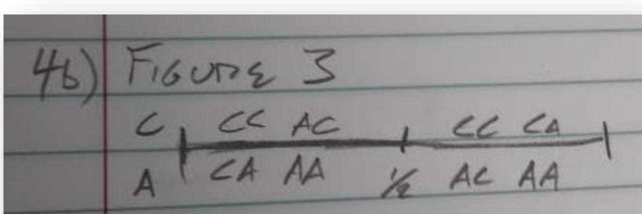
If Player 2 plays C_MC_V then Player 1's best response is to play C_M (because 3 > 2).

If Player 2 plays A_MA_V, then Player 1's best response is to play A_M (because 1 > 0).

Further, assuming Player 2 plays A_MC_V, Player 1 will respond with C_M only when:
$3-3p \geq 2-p$
$1 \geq 2p$
$p \leq ½$

Similarly, assuming Player 2 plays C_MA_V, Player 1 will respond with C_M only when:
$3p \geq p+1$
$2p \geq 1$
$p \geq ½$

It will thus be rational for the moderate player (Player 1) to cooperate against A_MA_V when $p \leq ½$, and to cooperate against C_MA_V when $p \geq ½$. We can represent Player 1's best responses on a number line as follows (Figure 3):



We must now investigate whether Player 2 has any incentive to deviate. Player 2 will always deviate from $A_M A_V$ to $C_M A_V$ because $3 > 2$. However, Player 2 has no incentive to respond to C with anything other than $C_M A_V$ where $p > ½$. (C, $C_M A_V$) is therefore a Bayesian Nash equilibrium. There's also a BNE at (C, $C_M C_V$) for $p = 1$ (Player 2 will be indifferent between $C_M C_V$ and $C_M A_V$).

$Br_2(A) =$ CA, AA when $p = 0$
AC, AA when $p = 1$
AA when $p = (0,1)$

$Br_2(C) =$ CA, AA when $p = 0$
CA, CC when $p = 1$
CA when $p = (0,1)$

If $p \in [1/2, 1]$, then (C, CA) is a Bayesian Nash equilibrium (the known moderate will cooperate).

If $p = 1$, then (C, CA) is BNE because Player 2 will no longer have an incentive to deviate to CA ($3 = 3$).

Also, if $p \in [0, 1]$ then (A, AA) is a BNE.

4c)

1\2	$C_M C_V$	$A_M A_V$	$C_M A_V$	$A_M A_V$
C_M	3	3-3p	3p	0
	p+2	2	3	3-p
A_M	2	2-p	p+1	1
	0	p	1-p	1
C_M	2	2-2p	2p	0
	p+2	2	3	3-p
A_M	3	3-2p	2p+1	1
	0	p	1-p	1

Players will need to update their beliefs after finding out their type. Types will be updated to the two cases below for Player 1 (and symmetrically for Player 2).

Case 1	$\theta_2 = V$	$\theta_2 = M$
$\theta_1 = M$	1/4	3/4

Case 2	$\theta_2 = V$	$\theta_2 = M$
$\theta_1 = V$	3/4	1/4

This means that both players (if moderate) can expect that their opponent will be moderate with a probability of ¾. Additionally, both players (if vainglorious) can expect that their opponent will be moderate with a probability of ¼. This results in the following utility functions:

$U_V(S_1, S_2) = [¾(U_V(S_1, S_{2V}))+ ¼U_V(S_1,S_{2M})]\ (½) + [¼(U_M(S_1,S_{2V})) + ¾[U_M(S_1, S_{2M})](½)$
$U_M(S_1,S_2) = [¾(U_M(S_1, S_{2M}))+ ¼U_M(S_1,S_{2V})]\ (½) + [¼(U_V(S_1,S_{2M})) + ¾[U_V(S_1, S_{2V})](½)$

1\2	C_MC_V	A_MC_V
C_MC_V	3*(3/8)+2*(1/8)+3*(1/8)+2*(3/8)	0*(3/8)+0*(1/8)+3*(1/8)+2*(3/8)
	3*(3/8)+3*(1/8)+2*(1/8)+2*(3/8)	2*(3/8)+2*(1/8)+2*(1/8)+2*(3/8)
A_MC_V	2*(3/8)+2*(1/8)+2*(1/8)+2*(3/8)	1*(3/8)+0*(1/8)+2*(1/8)+2*(3/8)
	0*(3/8)+0*(1/8)+3*(1/8)+2*(3/8)	1*(3/8)+0*(1/8)+2*(1/8)+2*(3/8)
C_MA_V	3*(3/8)+3*(1/8)+3*(1/8)+3*(3/8)	0*(3/8)+1*(1/8)+3*(1/8)+3*(3/8)
	3*(3/8)+2*(1/8)+0*(1/8)+0*(3/8)	2*(3/8)+2*(1/8)+1*(1/8)+0*(3/8)
A_MA_V	2*(3/8)+3*(1/8)+2*(1/8)+3*(3/8)	1*(3/8)+1*(1/8)+2*(1/8)+3*(3/8)
	0*(3/8)+0*(1/8)+0*(1/8)+0*(3/8)	1*(3/8)+0*(1/8)+1*(1/8)+0*(3/8)

1\2	C_MA_V	A_MA_V
C_MC_V	3*(3/8)+2*(1/8)+0*(1/8)+0*(3/8)	0*(3/8)+0*(1/8)+0*(1/8)+0*(3/8)
	3*(3/8)+3*(1/8)+3*(1/8)+3*(3/8)	2*(3/8)+3*(1/8)+2*(1/8)+3*(3/8)
A_MC_V	2*(3/8)+2*(1/8)+1*(1/8)+0*(3/8)	1*(3/8)+1*(1/8)+0*(1/8)+0*(3/8)
	0*(3/8)+1*(1/8)+3*(1/8)+3*(3/8)	1*(3/8)+1*(1/8)+2*(1/8)+3*(3/8)
C_MA_V	3*(3/8)+3*(1/8)+0*(1/8)+1*(3/8)	0*(3/8)+1*(1/8)+0*(1/8)+1*(3/8)
	3*(3/8)+3*(1/8)+0*(1/8)+1*(3/8)	2*(3/8)+3*(1/8)+1*(1/8)+1*(3/8)
A_MA_V	2*(3/8)+3*(1/8)+1*(1/8)+1*(3/8)	1*(3/8)+1*(1/8)+1*(1/8)+1*(3/8)
	0*(3/8)+1*(1/8)+0*(1/8)+1*(3/8)	1*(3/8)+1*(1/8)+1*(1/8)+1*(3/8)

We will now plug in p-values of 3/8 (moderate type) and 1/8 (vainglorious type). This results in the following table:

1\2	C_MC_V	A_MC_V	C_MA_V	A_MA_V
C_MC_V	2 1/2	1 1/8	1 3/8	0
	2 1/2	2	3	2 1/2
A_MC_V	2	1 3/8	1 1/8	1/2
	1 1/8	1 3/8	1 5/8	1 7/8
C_MA_V	3	1 5/8	**1 7/8**	1/2
	1 3/8	1 1/8	**1 7/8**	1 5/8
A_MA_V	2 1/2	1 7/8	1 5/8	**1**
	0	1/2	1/2	**1**

Simplifying, we get:

1\2	C_MC_V	A_MC_V	C_MA_V	A_MA_V
C_MC_V	5/2, 5/2	9/8, 2	11/8, 3	0, 5/2
A_MC_V	2, 9/8	11/8, 11/8	9/8, 13/8	1/2, 15/8
C_MA_V	3, 11/8	13/8, 9/8	**15/8, 15/8**	1/2, 13/8
A_MA_V	5/2, 0	15/8, 1/2	13/8, 1/2	**1, 1**

4d) This game has two pure-strategy BNEs (in bold, above). In ($A_M A_V$, $A_M A_V$) both players attack. Vainglorious players have a strict dominant strategy to attack (strategies involving C_V are strictly dominated). In ($C_M A_V$, $C_M A_V$), if one player is a moderate, they perceive that their opponent may be moderate with a probability of ¾, and consequently the expected value of cooperation (15/8) is higher than the expected value of attacking (1). These are the only pure-strategy BNEs in this game because each player is acting based on the other player's possible types, of which there are two. Hobbes' state of nature is represented by ($A_M A_V$, $A_M A_V$). In ($C_M A_V$, $C_M A_V$), vainglorious types attack while moderate types cooperate.

4e) The equilibrium at ($C_M A_V$, $C_M A_V$) is Pareto superior to the equilibrium at ($A_M A_V$, $A_M A_V$) because $15/8 > 1$. This tracks our intuition – gains are higher when there is the possibility of cooperation. As this is a symmetrical game, equilibria will be found in cases where players are playing parallel strategies (along the diagonal). This leaves two equilibria to consider - ($C_V A_M$, $C_V A_M$) and ($C_M C_V$, $C_M C_V$). Neither of these cases will result in Nash equilibria. ($A_M C_V$, $A_M C_V$) guarantees that players will have at least one profitable deviation because each player's type yields a higher payoff for the opposite action. Similarly the case of ($C_M C_V$, $C_M C_V$) will guarantee that players playing C_V have a profitable deviation to an off-type strategy. In general, it seems that strategies requiring particular types to play non-preferred actions will never be subgame perfect because there will always be profitable deviations to type-consistent behavior with higher payoffs.

7a) First, we will define the players' (symmetric) utility functions:

$$U_1(e_1, e_2, e_3; \theta_1) = \begin{cases} \theta^2 - c & \text{if } e_i = 1 \\ \theta^2 & \text{if } e_i = 0, e_j = 1, e_k = 1 \\ \theta^2 & \text{if } e_i = 0, e_j = 0, e_k = 1 \\ \theta^2 & \text{if } e_i = 0, e_j = 1, e_k = 0 \\ 0 & \text{if } e_i = e_j = e_k = 0 \end{cases}$$

The inequality we wish to solve is:

$$u_1(e_1 = 1, e_2, e_3; \theta_1) \geq u_1(e_1 = 0, e_2, e_3; \theta_1)$$

Expected utility for either action is a function of Player 1's type and the other players' strategies, so:
$u_1(e_1 = 1, e_2, e_3; \theta_1) = \theta^2 - c$

We can rewrite our inequality as:

$$\theta^2 - c \geq \Pr(e_2 = 1, e_3 = 1 \mid \theta_1)\, \theta^2 + \Pr(e_2 = 1, e_3 = 0 \mid \theta_1)\theta^2 + \Pr(e_2 = 0, e_3 = 1 \mid \theta_1)\theta^2$$

Since $\Pr(e_2 = 1, e_3 = 1 \mid \theta_1) = \Pr(e_2 = 1, e_3 = 1)$

then we factor out θ^2 from the right side:

$$\theta^2 - c \geq \Pr(e_2 = 1, e_3 = 1 \mid \theta_1) + \Pr(e_2 = 1, e_3 = 0 \mid \theta_1) + \Pr(e_2 = 0, e_3 = 1 \mid \theta_1)$$

$$c \leq \theta^2(1 - \Pr(e_2 = 1, e_3 = 1 \mid \theta_1) + \Pr(e_2 = 1, e_3 = 0 \mid \theta_1) + \Pr(e_2 = 0, e_3 = 1 \mid \theta_1))$$

$$\theta \geq \sqrt{}(c/(1 - \Pr(e_2 = 1, e_3 = 1 \mid \theta_1) + \Pr(e_2 = 1, e_3 = 0 \mid \theta_1) + \Pr(e_2 = 0, e_3 = 1 \mid \theta_1)) = \hat{\theta}$$

We will now sum out the joint probabilities (conjuncts are independent):

$\theta \geq \surd(c/(1 - ((Pr(e_2 = 1))(Pr(e_3 = 1)) + ((Pr(e_2 = 1))(Pr(e_3 = 0)) + ((Pr(e_2 = 0))(Pr(e_3 = 1))$

This can be rearranged to:

$\theta \geq \surd(c/(1 - ((Pr(e_2 = 1))(Pr(e_3 = 1)) + ((Pr(e_2 = 1))(1 - Pr(e_3 = 1)) + ((1 - Pr(e_2 = 1))(Pr(e_3 = 1))$

Since types are symmetrical:

$\hat{\theta}_i \geq \surd(c/ (\theta_j)(\theta_k)$

Now we have Player 1's cut-point strategy – they will choose e1 = 1 if $\theta_1 \geq \hat{\theta}_1$ and e1 = 0 otherwise. Each player's cut-point is a function of their beliefs about the probability that the other players will invest effort.

We can now define the players' threshold types:

$\hat{\theta} = \hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}_3$

Since all players know that each player's types are uniformly distributed on the unit interval and players will invest if their type is above the threshold type, we can conclude that:

$1 - F(\hat{\theta}_i) = 1 - \hat{\theta}_i$

This means that all players believe that other players will invest effort with probability $1 - \hat{\theta}_i$

Substituting this into our cut-point strategy, we get:

$Pr(e_1 = 0) = (1 - \hat{\theta}_i)$

$\hat{\theta}_1 = \surd(c/ (\hat{\theta}_2)(\hat{\theta}_3))$
$\hat{\theta}_2 = \surd(c/ (\hat{\theta}_1)(\hat{\theta}_3))$
$\hat{\theta}_3 = \surd(c/ (\hat{\theta}_1)(\hat{\theta}_2))$

The players' threshold rules will therefore be as follows:

$\hat{\theta}_1 (\hat{\theta}_2)(\hat{\theta}_3) = \surd(c/ \hat{\theta}_2 \hat{\theta}_3)$ if $\hat{\theta}_2 \hat{\theta}_3 \geq c$; 1 otherwise
$\hat{\theta}_2 (\hat{\theta}_1)(\hat{\theta}_3) = \surd(c/ \hat{\theta}_1 \hat{\theta}_3)$ if $\hat{\theta}_1 \hat{\theta}_3 \geq c$; 1 otherwise
$\hat{\theta}_3 (\hat{\theta}_1)(\hat{\theta}_2) = \surd(c/ \hat{\theta}_1 \hat{\theta}_2)$ if $\hat{\theta}_1 \hat{\theta}_2 \geq c$; 1 otherwise

We can refine this further as follows:

$\hat{\theta}_1 = \surd(c/ (\hat{\theta}_2)(\hat{\theta}_3))$
$(\hat{\theta}_1)^2 = c/ (\hat{\theta}_2)(\hat{\theta}_3)$
$c = (\hat{\theta}_1)^2 (\hat{\theta}_2) (\hat{\theta}_3)$

Since all three players are symmetrical, we know:

$c = (\hat{\theta}_1)^2 (\hat{\theta}_2)(\hat{\theta}_3)$
$c = (\hat{\theta}_2)^2 (\hat{\theta}_1)(\hat{\theta}_3)$
$c = (\hat{\theta}_3)^2 (\hat{\theta}_1)(\hat{\theta}_2)$

As a result, we can conclude that:

$c = (\hat{\theta}_1)^2 (\hat{\theta}_2) (\hat{\theta}_3) = (\hat{\theta}_2)^2 (\hat{\theta}_1) (\hat{\theta}_3) = (\hat{\theta}_3)^2 (\hat{\theta}_1) (\hat{\theta}_2)$

Now we divide through by $((\hat{\theta}_1) (\hat{\theta}_2) (\hat{\theta}_3))$, which gives us:

$(\hat{\theta}_1) = (\hat{\theta}_2) = (\hat{\theta}_3)$

We can now restate the players' (identical) threshold rules as follows:

$\theta = \sqrt{(c/ (\hat{\theta})(\hat{\theta}))}$

$(\hat{\theta})^2 = c/ (\hat{\theta})^2$

$(\hat{\theta})^2 ((\hat{\theta})^2) = c/ (\hat{\theta})^2 (\hat{\theta})^2$

$c = (\hat{\theta})^4$

$\hat{\theta} = c^{1/4}$

If the product of the other two players' threshold types is $\geq$ c, then for all players:

$\hat{\theta}_i (\hat{\theta}_j)(\hat{\theta}_k) = c^{1/4}$ if $(\hat{\theta}_j)(\hat{\theta}_k) \geq c$; 1 otherwise

b) The equilibrium strategy for this game is:

$s_i^*(\hat{\theta}_i) =$ $(e_i = 0)$ if $\theta_i < c^{1/4}$;
$(e_i = 1)$ if $\theta_i \geq c^{1/4}$

c) This allows us to state a Bayesian Nash equilibrium for this game where all players are following the equilibrium strategy:

Claim 1: If $(\theta_i) \geq c$ for i = {1, 2, 3}, then the following strategy profile is a Bayesian Nash equilibrium:

Player i's best response should be to contribute no effort if her type is below the threshold type, and to contribute if her type is equal to or above the threshold type.

If all players' types are $\geq$ c, then $\hat{\theta}$ must be > c, because $\hat{\theta} = c^{1/4}$ and $\hat{\theta}$ and c are $\in [0, 1]$.

We can prove that $\hat{\theta}$ will always be > c. Assume that $\hat{\theta} < c$. We know $\hat{\theta}_i = c^{1/4}$, and we know that $\hat{\theta}$ and c are $\in [0, 1]$. The nth root of any number between 0 and 1 will always be larger than the original number (n > 1). This means that $c^{1/4} > c$. When combined with our assumption, this results in a contradiction.

A reflection on the problem-solving process

The process is slowly becoming easier. I'm focusing more on practice problems, and I'm now starting with the practice problems and returning to the text to help me with particular solution concepts. I've been doing the other problems in the problem sets as practice, as well as problems out of Tadelis and Watson.

a) Initially, I solved the problems on my own between 11/16 and 11/18. Then, I reviewed approaches and other strategies with Patrick, Adam, Tom, Lauren, Terry, Kevin and Bianca.

b) I used Tadelis extensively throughout the problem-solving process. As usual, I relied extensively on the class notes, particularly in my solution to #7. While typesetting, I consulted Google extensively.

c) The main thing I need is more practice with static games of incomplete information. I'd like to see a few more worked examples with approved solutions so as to work my own way through them and follow the reasoning.

Game Theory Problem Set 5

Matthew Draper

12/7/18

2. Since both players have four possible strategies, there are a total of sixteen pure strategy profiles that we need to check for perfect Bayesian equilibria. They are: {AAhh}, {ABhh}, {BAhh}, {BBhh}, {AAhl}, {ABhl}, {BAhl}, {BBhl}, {AAlh}, {ABlh}, {BAlh}, {BBlh}, {AAll}, {ABll}, {BAll}, {BBll}.

To begin, we will determine Player 2's best responses at each information set.

First we consider Player 2's expected utility when Player 1 as type A:

$EU_2(h,A)$: $q(0) + (1-q)(1) \geq q(1) + (1-q)(0)$

$q \leq ½$

$EU_2(l,A)$: $q(1) + (1-q)(0) \geq q(0) + (1-q)(1)$

$q \geq ½$

BR_2 (q) = if $q < ½$, {h}

if $q = ½$, {l,h}

if $q > ½$, {l}

Similarly for the case of Player 1 as type B:

$EU_2(h,B)$: $r(0) + (1-r)(1) \geq r(1) + (1-r)(0)$

$r \leq ½$

$EU_2(l,B)$: $r(1) + (1-r)(0) \geq r(0) + (1-r)(1)$

$r \geq ½$

BR_2 (r) = {h} if $r < ½$

{l,h} if $r = ½$

{l} if $r > ½$

Now we will consider the possible pooling equilibria (where it's rational for Player 1 to play the same strategy regardless of type). There are two – s1 = {AA} or {BB}.

First we will consider {AA}. If Player 1 plays {AA}, then no updating is possible for Player 2, so we know that $q = p = ¾$. The probability of Player 1 playing {B} will be 0, so r will be undefined (0 in the denominator of Bayes' rule). Since $q = ¾$, we know by reference to their best response function that Player 2 will respond with {l} on the path of play, and we can't say anything yet about their off-the-path strategy since r is undefined.

Now we will consider {BB}. If Player 1 plays {BB}, then no updating is possible for Player 2, so we know that $r = ¾$. The probability of Player 1 playing {A} will be 0, so q will be undefined. Since $r = ¾$, we know that Player 2 will respond with {l} on the path of play, and we can't say anything yet about their off-the-path strategy since q is undefined.

The foregoing analysis allows us to rule out a number of pooling equilibria – those where Player 2 employs a strategy inconsistent with the best responses outlined above. The potential pooling equilibria are: {AAhh}, {AAhl}, {AAlh}, {AAll}, {BBhh}, {BBhl}, {BBlh}, {BBll}. For s1 = AA and s1 =

BB, we can eliminate any equilibria where Player 2 plays h on the path of play. This rules out four of our eight candidates, leaving us with: {AAlh}, {AAll}, {BBhl} and {BBll}.

Now we will consider separating equilibria.

First, we will consider $s_1 = \{A,B\}$. In this strategy Type A will play {A} and Type B will play {B} with a one-to-one correspondence between types and actions. Although Player 2 doesn't know Player 1's type, they do know Player 1's type-dependent strategy, which means that they know that Player 1 will play A if and only if they are type A, and similarly will play B if and only if they are type B. So if Player 2 sees A played, they will update their beliefs and will consequently know that Player 1 is Type A and that $q = 1$ and $r = 0$. If they see B played, they will likewise update their beliefs and will know that Player 1 is Type B and that $q = 0$ and $r = 1$. Given these values of q and r, Player 2's best response function indicates that they will respond by playing {l} in response to A and {h} in response to B.

Now we will consider $s_1 = \{B,A\}$. In this strategy Type A will play {B} and Type B will play {A} with a one-to-one correspondence between types and actions. Although Player 2 doesn't know Player 1's type, they do know Player 1's type-dependent strategy, which means that they know that Player 1 will play B if and only if they are type A, and similarly will play A if and only if they are type B. So if Player 2 sees A played, they will update their beliefs and will consequently know that Player 1 is Type B and that $q = 1$ and $r = 0$. If they see B played, they will likewise update their beliefs and will know that Player 1 is Type A and that $q = 0$ and $r = 1$. Given these values of q and r, Player 2 will respond by playing {h} in response to A and {l} in response to B.

The foregoing analysis specifies one unique rational strategy profile for Player 2 in response to each of {A,B} and {B,A}. In the case of {A,B}, Player 2's only rational response is {l,h} and Player 2's only rational response to {B,A} is {h,l}. We can therefore eliminate six of the eight potential separating equilibria, leaving only {ABlh} and {BAhl}.

We will now evaluate our remaining six potential perfect Bayesian equilibria. In each case, we have demonstrated that Player 2 is playing a best response to Player 1's strategy. We must now check each of our six candidates to determine whether Player 1 is playing a best response to Player 2's strategy using the method of checking for single rational deviations.

Separating Equilibria

First, consider {ABlh}. Player 1 has no single rational deviations ($2 > 0$, $1 = 1$).

Now consider {BAhl}. Player 1 has no single rational deviations ($2 > 0$, $1 = 1$).

Pooling Equilibria

Now consider {AAlh}. Player 1 has no single rational deviations ($2 > 0$, $1 = 1$).

Now consider {AAll}. Player 1 has no single rational deviations ($2 > 0$, $1 = 1$).

Now consider {BBhl}. Player 1 has an incentive to deviate as the B-type ($2 > 0$), so this is not PBE.

Now consider {BBll}. Player 1 has an incentive to deviate as both types ($1 > 0$, $2 > 1$) so this is not PBE.

We have now considered all 16 pure-strategy potential equilibria, and have sustained four of them as perfect Bayesian equilibria. They are: {ABlh}, {BAhl}, {AAlh}, {AAll}.

Semi-separating equilibria

We must now consider mixed-strategies and semi-separating equilibria. We will re-examine our pooling equilibria to find off-the-path strategies that keep Player 1 indifferent between their pure strategies and therefore open to the possibility of employing a mixed strategy.

There are four strategic possibilities for Player 2 – s_2 = {hh}, {hl}, {lh} and {ll}. We will examine each in turn. In the case of {hh}, Player 1 will have an incentive to play A at both types ($2 > 1$, $1 > 0$). In the case of {hl}, Player 1's Type A will play A but Type B will be indifferent ($2 > 0$, $1 = 1$). In the case of {lh}, the situation is reversed and Type A will be indifferent ($1 = 1$, $2 > 0$). Finally, in the case of {ll}, Player 1 will have an incentive to play A at both types ($2 > 1$, $1 > 0$).

We therefore have potential separating equilibria only at s_2 = {hl}, {lh}.

We will now attempt to confirm our results using the normal-form method. We will construct a normal-form table and identify the pure-strategy Nash equilibria.

1\2	hh'	hl'	lh'	ll'
$A_A A_B$	5/4, 1/4	5/4, 1/4	**7/4, 3/4**	**7/4, 3/4**
$A_A B_B$	1, 1/4	3/4, 0	**7/4, 1**	3/2, 3/4
$B_A A_B$	1/2, 1/4	**5/4, 1**	1/4, 0	1, 3/4
$B_A B_B$	1/4, 1/4	3/4, 3/4	1/4, 1/4	3/4, 3/4

Pure-strategy Nash equilibria:

($A_A B_B$, lh'), ($B_A A_B$, hl'), ($A_A A_B$, lh'), ($A_A A_B$, ll')

Now we will check the pure-strategy Nash equilibria for perfect Bayesian equilibrium.

($A_A A_B$, lh') is PBE as long as $q = ¾$ (no updating is possible, so $q = p$), and $r \leq ½$ ($1-(1-r) + 0r \geq 0(1-r) + 1r$; $1-r \geq r$; $½ \geq r$) (probability of being at r's information set is 0, so Bayes' rule returns an undefined result. There is no single profitable deviation for either player.) Since $q > ½$, our best response function dictates that player 2 will play {l} in response to {A}. Since r is undefined, we can't say for sure that Player 2's strategy will contain h'. This is a pooling equilibrium.

The same analysis applies to ($A_A A_B$, ll'). ($A_A A_B$, ll') is PBE as long as $q = ¾$ and $r \geq ½$ ($1-(1-r) + 0r \geq 0(1-r) + 1r$; $1-r \geq r$; $½ \geq r$) (probability of being at r's information set is 0, so Bayes' rule returns an undefined result. There is no single profitable deviation for either player.)

($A_A B_B$, lh') is PBE as long as $q = 1$ and $r = 0$. There is no single profitable deviation for either player.

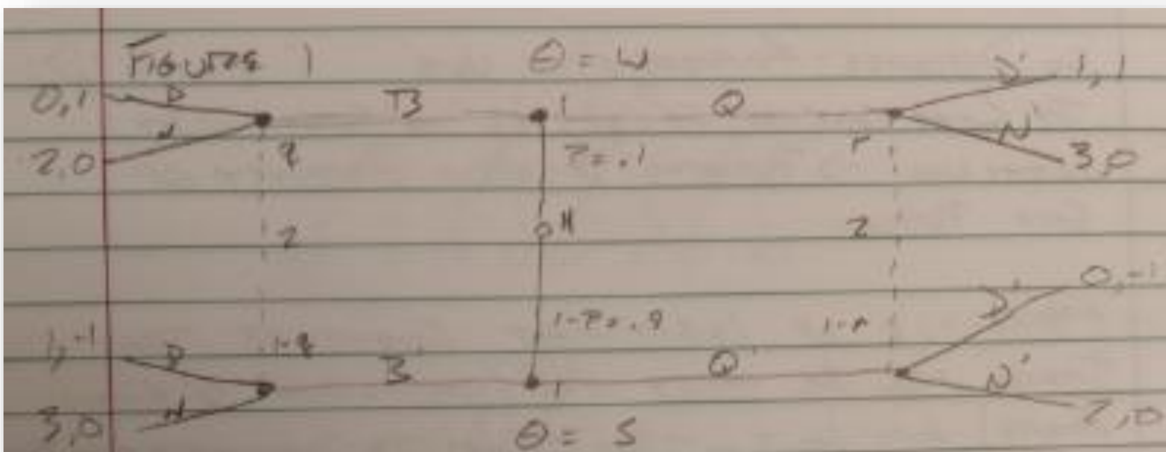
($B_A A_B$, lh') is PBE as long as $q = 0$ and $r = 1$. There is no single profitable deviation for either player.

Now we will check for mixed strategies. Row $B_A B_B$ and column hh' are strictly dominated.

1\2	~~hh'~~	hl'	lh'	ll'
A_AA_B	~~5/4, 1/4~~	5/4, 1/4	**7/4, 3/4**	**7/4, 3/4**
A_AB_B	~~1, 1/4~~	3/4, 0	**7/4, 1**	3/2, 3/4
B_AA_B	~~1/2, 1/4~~	**5/4, 1**	1/4, 0	1, 3/4
~~B_AB_B~~	~~1/4, 1/4~~	~~3/4, 3/4~~	~~1/4, ¼~~	~~3/4, 3/4~~

There is additional weak dominance, but no further strict dominance. Now assume that Player 1 plays a mixed strategy with p = .5 on {AB} and (1-p) = .5 on {BA}. There will be equilibrium at either hl or lh, because q = r = ½, so by the best response functions above Player 2 will be indifferent between playing l and h. This is a semi-separating equilibrium.

7a) See Figure 1.



b) We will now construct the probability-weighted normal-form matrix. Pure-strategy Nash equilibria are in bold.

1\2	dd'	dn'	nd'	nn'
B_WB_S	.9, -.8	.9, -.8	**2.9*, 0***	2.9, 0*
B_WQ_S	0, -.8	1.8, .1*	.2, -.9	2, 0
Q_WB_S	1*, -.8	1.2, -.9	2.8, .1*	3*, 0
Q_WQ_S	.1, -.8	**2.1*, 0***	.1, -.8	2.1, 0*

c) This gives us two pure-strategy Nash equilibria, at {BBnd'} and {QQdn'}. To sustain these as Bayesian Nash equilibria, we will need to show that each player's type-contingent strategy is a best response to opponents' type-contingent strategies, given players' beliefs about types. In this example, only Player 1 has types. Consequently, we wish to show that Player 2's type-contingent strategy is a best-response to Player 1's pure and mixed strategies.

First we will determine Player 2's best response function:

EU (d|B) => $q(1)+(1-q)(-1) \geq q(0) + (1-q)(0); q \geq ½$

EU (n|B) => $q(0) + (1-q)(0) \geq q(1)+(1-q)(-1); q \leq ½$

EU (d|Q) => $r(1)+(1-r)(-1) \geq r(0) + (1-r)(0); r \geq ½$

EU (n|Q) => $r(0) + (1-r)(0) \geq r(1)+(1-r)(-1); r \leq ½$

BR_2 (r) = {n} if r < ½

{n,d} if r = ½

{d} if r > ½

BR_2 (q) = {n} if q < ½

{n,d} if q = ½

{d} if q > ½

These are best responses to Player 1's pure strategies.

We will now consider mixed strategies. When mixing, we will eliminate possibilities that are strictly dominated. We can eliminate dd' because it is strictly dominated by nn'. We can then eliminate BwQs, because it is strictly dominated by a mix of BB and QQ, with p on BB and 1-p on QQ, as long as p is between 1/28 and 1/4.

$0.3 > 1.2p; 1/4 > p$

$0.9(p) + 2.1 (1-p) > 1.8$, and $2.9p + 0.1(1-p) > 0.2$

So this mixed strategy will be sustained when p > 1/28 and p < ¼. Player 2 will respond with nd' because r < ½. This gives us a mixed strategy equilibrium at {σ = p > 1/28 and p < ¼, nd'}.

The eliminated rows are reflected on the table below:

1\2	~~dd'~~	dn'	nd'	nn'
$B_W B_S$	~~.9, -.8~~	.9, -.8	2.9, 0	2.9, 0
~~$B_W Q_S$~~	~~0, -.8~~	~~1.8, .1~~	~~.2, -.9~~	~~2, 0~~
$Q_W B_S$	~~1, -.8~~	1.2, -.9	2.8, .1	3, 0
$Q_W Q_S$	~~.1, -.8~~	2.1, 0	.1, -.8	2.1, 0

d) Now we will evaluate strategy profiles for perfect Bayesian equilibrium.

We will conjecture a PBE at QQdn'. By our best response functions, r = 0.1 so Player 2 will respond to QQ with {dn'} as long as q ≥ ½, which sustains it as PBE. This is a pooling equilibrium.

We will now conjecture a PBE at BBnd'. By our best response functions, $q = 0.1$ Player 2 will respond to BB with n' as long as $r > ½$, which sustains it as PBE. This is a pooling equilibrium.

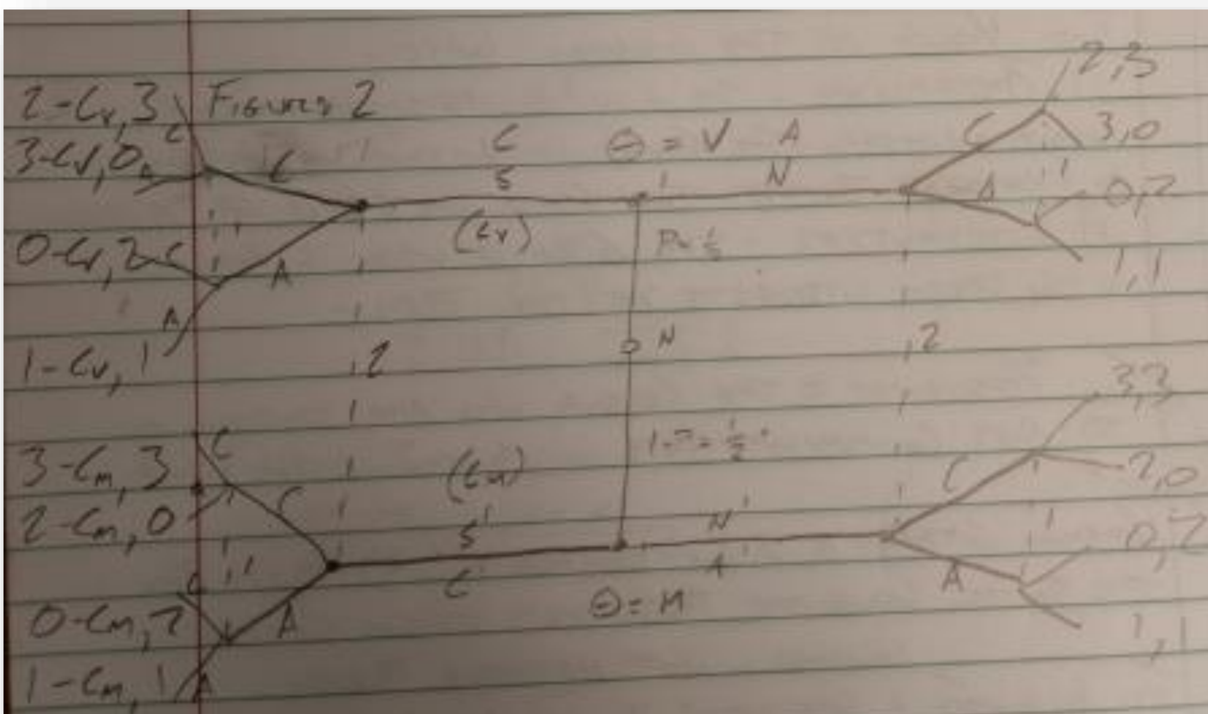
We will now conjecture a mixed strategy where Player 1 mixes between QwBs (s) and QwQs (1-s) and Player 2 mixes between nd' and nn' with a value of t on nd' and a value of 1-t on nn'. We can use Bayes' rule to determine Pr(W|Q) as below:

$Pr(W|Q) = (Pr(Q|W)Pr(W))/Pr(Q|W)Pr(W)+Pr(Q|S)+Pr(S) = 0.1/.1(1-s)(0.9) = 1/(10-9s)$. Since Player 1 is indifferent at $r = ½$, we can set the solution above equal to ½, which results in a value of $s = 8/9$. This is a semi-separating equilibrium.

e) The mixed strategy defined above is not a perfect Bayesian equilibrium. {$\sigma = p > 1/28$ and $p <$ ¼, nd'} because requirement #3 is violated – off-the-path information sets are not consistent with Bayes' rule (Player 2's best response does not take updated beliefs into account).

f) There are no separating equilibria because in either case ($s_1 =$ {BB}, {QQ}), at least one of Player 1's types will have an incentive to deviate. In the case of $s_1 =$ {BB}, $q = p = .1$, so by the best response functions derived above, Player 2 will play {nn'}. Player 1's W type will then have an incentive to deviate to Q ($3 > 2$). Similarly in the case of $s_1 =$ {QQ}, $r = p = .1$, so Player 2 will play {dd'} and Player 1's W type will have an incentive to deviate ($1 > 0$).

8a) See Figure 2



b) We will first consider pooling equilibria. We need only consider $\Theta_1 = V$ profiles where Player 1 chooses to attack, as all other profiles are strictly dominated. Let us begin with profiles where Player 1 does not send the signal. We will begin with NAANAA. Consider the conjectured strategy $\sigma_1 = NAA\ (\Theta_1=V)\ NAA\ (\Theta_1=M)$. Player 2's best response will be to play {A} for any value of r. Q can be any number between 0 and 1. There is thus a potential equilibrium at (NAANAA,AA). We must check for potential deviations by Player 1. Player 1 has no incentives to deviate at the final nodes (A is best in all 4 cases), and has no incentive to deviate at their two initial nodes. This is a pooling equilibrium. Values of C_V and C_M may be any positive integer.

The other possibilities are σ_1 = NAANAC and σ_1 = NAANCC. We can rule out NAANCA because Player 1 has a profitable deviation (to NAA for $\Theta_1 = M$). Looking at NAANAC, we see that Player 2's best response will be to choose {A} if $q > ½$ and {C} if $q < ½$, and to choose {A} if $r > ½$ and {C} if $r < ½$. But if Player 2 plays these best responses, $\Theta_1 = V$ will have an incentive to deviate. This rules it out as an equilibrium. In the case of NAANCC, Player 2's best response is {c} for all values of q and r. Since this is common knowledge, both $\Theta_1 = M$ and $\Theta_1 = V$ will have an incentive to deviate, ruling this out as an equilibrium.

Now we must consider cases where Player 1 sends the signal. Recall that situations where $\Theta_1 = V$ cooperates are ruled out by strict dominance. We must therefore begin with SAA, and the $\Theta_1 = M$ type has four options, giving us σ_1 = SAASAA, SAASAC, SAASCA and SAASCC. All four situations have the moderate type sending a costly signal, so we hypothesize that strategy profiles where $\Theta_1 = M$ sends the signal and then attacks are unlikely to result in equilibria. We will begin with the least plausible candidates.

Consider σ_1 = SAASAA. Player 2's best response will be to play {C} for all values of r and {A} for all values of q. Knowing this, Player 1 would benefit from switching to N. The same logic applies to SAASCC. In the case of SAASAC, Player 2's best response will be to play {C} if $q < ½$, to play {A} if $q > ½$, and indifference at $q = ½$. Knowing this, Player 1 will have an incentive to deviate to {C}, ruling it out as an equilibrium.

Now consider σ_1 = SAASCA. Player 2' best response will be to play {A} if $q > ½$, {C} if $q < ½$, and indifference if $q = ½$ ($U2(C,q) \geq U2(A,q)$, which is $3\text{-}3q \geq q + 2 - 2q$, or $½ \geq q$). Despite knowing this, Player 1 will have no incentive to deviate as long as $3\text{-}C_V > 1$ and $2 > C_V$. R must be between 0 and 1, and C_M and C_V may be any positive integer. Subject to those conditions, SAASCA,AC is a pooling equilibrium.

c) We will now consider separating equilibria. We need to consider profiles where one type signals and the other does not. This provides for a very large number of cases. We will consider likely candidates. The likeliest profiles to result in equilibria are where the types take actions consonant with their natures (because deviation is less likely as payoffs are higher), which means that $\Theta_1 = V$ will attack and $\Theta_1 = M$ will cooperate. Consider NAASCC. Player 2's best response will be to choose {A} as long as $r \geq ½$, and to choose {C} if $q > ½$. This gives $\Theta_1 = M$ a profitable deviation to N, but this profile seems close to equilibrium. It's likely that we will find an equilibrium by holding NAA constant and adjusting the other side, so we will consider NAASAC and NAASCA.

σ_1 = NAASAC can be ruled out for the same reasons we ruled out SAASAC, so we will consider σ_1 = NAASCA. Player 2's best responses will be to play {A} for all values of q and r. We will now check for possible deviations by both of Player 1's types. $\Theta_1 = V$ does not have an incentive to deviate within the subgame because $3 > 2$ and $1 > 0$, and does not have an incentive to deviate at the root because

$3 - C_V > 2 - C_V$ and $1 - C_V > 0 - C_V$ for all nonnegative values of C_M and C_V. $\Theta_1 = M$ does not have an incentive to deviate within the subgame because $1 - C_M > 0 - C_M$ and $3 - C_M > 2 - C_M$ for all nonnegative values of C_M and C_V. SAASAC,AA is therefore a separating equilibrium.

d) We will now consider semi-separating equilibria where $\Theta_1 = M$ or $\Theta_1 = V$ mixes at their initial node. We will first consider a mix by $\Theta_1 = M$. We must consider Player 2's best responses.

$EU_2(C,q) \geq EU_2(A,q)$

$3(1 - q) \geq 1q + 2(1 - q)$

$\frac{1}{2} \geq q$

$BR_2(q) =$ {A} if $q > \frac{1}{2}$

{C} if $q < \frac{1}{2}$

{A,C} if $q = \frac{1}{2}$

For now, r may be any value $\in [0,1]$

$BR_2(r) =$ {A}

Conjecture the strategy profile $\sigma_1 = \sigma$AASCA, where Player 1 mixes at $\Theta_1 = V$ with a probability of t on S and 1 – t on N. In this case, Player 2 will update beliefs when Player 1 makes their initial move. By Bayes' rule:

$Pr(V|S) = (Pr(S|V)Pr(V) / Pr(S|V)Pr(V) + Pr(S|M)Pr(M))$

$Pr(V|S) = t(1/2) / t(1/2) + \frac{1}{2} = t / (t + 1)$

This equilibrium can be sustained as long as Player 1 has no incentive to deviate, which will be the case as long as 3 – CV > 1 and 3 – CM > 1, which is equivalent to CV < 2 and CM < 2. Q will equal t / (t + 1), but this is a knife-edge result because r must equal 1 to prevent the possibility of deviation by Player 2.

A reflection on the problem-solving process

I really enjoyed this material, and I think I'm beginning to have an ok grasp of dynamic games of incomplete information. Problem #8 is really, really hard, but I felt ok on the other two. I'm sad the course is ending, for two reasons – our tools are finally robust enough to start applying them to real problems, and the material is starting to get really fun. It's very satisfying to solve these problems.

a) Initially, I tackled the problems on my own between 11/30 and 12/3. Then, I reviewed approaches and other strategies with Patrick, Adam, Yi-Fan, Geoff, Tom, Lauren, Terry, Kevin and Bianca.

b) I used Tadelis extensively throughout the problem-solving process. As usual, I relied extensively on the class notes. While typesetting, I consulted Google extensively.

c) I just need more reps with everything. I wish we were continuing the course next quarter.