Problem Set #4

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POLI 211 - Formal Models in Political Theory

May 31, 2020

Exercise 1

Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Say that an alternative $x \in A$ is a *Condorcet winner* at a profile ρ if $|\{i \in N | x \succ_i y\}| \geq 2$.

Define the *majority rule choice function* f_M by $f_M(\rho) = x$ if and only if either (1) x is a Condorcet winner at ρ, or (2) no Condorcet winner exists and $x = a$ (you can think of a as the "default" alternative which is selected unless a Condorcet winner exists). Prove: f_M is not strategy-proof.

Proof. Suppose f_M is strategy-proof. This means that there exists no profile $\rho, \rho' \in \mathcal{P}^n$ such that for some individual $i \in N$, $f(\succ_i', \rho_-i) \succ_i f(\rho)$.

Now consider the following preferences:

1	2	3
a	b	c
b	c	b
c	a	a

Without loss of generality, consider the case of Player 3. If she reports her preferences accurately, there will be no Condorcet winner and a will be chosen as the "default alternative". By inaccurately reporting b as her first preference, Player 3 can ensure that the collective choice will be b, which she strictly prefers to a. This is a profile ρ where $f(\succ_i', \rho_-i) \succ_i f(\rho)$, so we must reject the conjecture with which we began. The majority rule choice function f_M must not be strategy-proof. □

Exercise 2

Say that a soical choice function is Pareto efficient if for any profile $\rho \in \mathcal{P}^n$ and any two alternatives $x, y \in A$, if $x \succ_i y$ for all $i \in N$, then $f(\rho) \neq y$. Define the *range* of a social choice function f as the following set: $f(\mathcal{P}^n) := \{x \in A | f(\rho) = x \text{ for some } \rho \in \mathcal{P}^n\}$.

Assume $|A| \geq 3$ and prove that if f is Pareto efficient, then $|f(\mathcal{P}^n)| \geq 3$.

Proof. Suppose that f is Pareto efficient. This implies that if all players strictly prefer x to y under a profile ρ, f cannot return y (stated generally, if all players prefer one alternative to another, the social choice function cannot return the alternative they do not prefer). Assume three alternatives, $\{x, y, z\} \in A$. We wish to prove that the range of f will be at least 3. That is equivalent to saying that the function has "full range," in the sense that for any alternative x, there exists some collection of revealed preferences ρ such that $f(\rho) = x$.[1]

Using the three alternatives $\{x, y, z\} \in A$, we must now show that there exist three profiles $\rho \in \mathcal{P}^n$ where the social choice function f selects each of the alternatives $\{x, y, z\}$ without violating Pareto efficiency. This condition means that in each case, there will be no alternative to the social choice produced by f which the players will strictly prefer. Taking the cases in turn, consider the following conjectured preferences (ρ^*):

1	2	3
x	x	z
y	y	y
z	z	x

In this case, there are no alternatives in A such that $x \succ_i y$ for all $i \in N$. Notice that when applied to these preferences, f returns the social choice of x. Recall that we defined the range of a social choice function f as the count of alternatives x for which there exists some collection of revealed preferences ρ such that $f(\rho) = x$. If we apply f to the conjectured ρ^*, it will return x. Because $y \not\succ_i x$ by all players, f is Pareto efficient and x is part of the function's range. It is trivial to repeat this exercise for y and z, showing that $|f(\mathcal{P}^n)|$ is at least 3.

□

[1] This property of restricting a function's codomain to the image of its domain is known as a "surjection" (see also "onto").

Exercise 3

Assume $A = \{a, b\}$ and $N = \{1, 2, 3\}$. Define the majority rule choice function as the rule $f_M : \mathcal{P}^n \to A$ for which $f(\rho) = a$ if and only if $|\{i \in N | a \succ_i b\}| > |\{i \in N | b \succ_i a\}|$. Prove that f_M is strategy-proof.

Proof. Recall that a social choice function $f : \mathcal{P}^n \to A$ is strategy-proof unless there are profiles $\rho, \rho' \in \mathcal{P}^n$ such that for some individual $i \in N$, $f(\succ_i', \rho_-i) \succ_i f(\rho)$. We therefore wish to show that it is never in a player's best interests to report inaccurate preferences.

Without loss of generality, consider Player 3's choice. There are four possible configurations of the preferences of Player 1 and Player 2 which Player 3 might confront:

1	2	3	1	2	3	1	2	3	1	2	3
x	x		y	y		x	y		y	x	
y	y		x	x		y	x		x	y	

Closer inspection reveals that these four cases collapse into two cases, with the labels of the preferences reversed. In the first, Players 1 and 2 have identical preferences, while in the second, they have symmetrical, opposite preferences. I will consider each of these in turn.

Suppose, without loss of generality, that Player 3's true preferences are given by $\{x \succ_3 y\}$. We can see that Player 3's choice will be irrelevant to the social choice in the first two cases above, because a majority already prefers x or y. In cases 3 and 4, true preference revelation by Player 3 will result in the social choice she prefers, because it will result in a majority for x. She can do no better (indeed, will do worse) by reporting a preference for y in any of the cases above. This indicates that there exists no profile $\rho, \rho' \in \mathcal{P}^n$ such that for some individual $i \in N$, $f(\succ_i', \rho_-i) \succ_i f(\rho)$. The social choice function f_M is therefore strategy-proof.

□