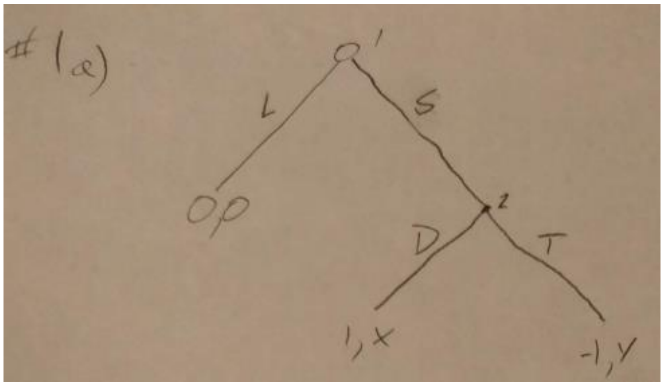
Game Theory Exam 2

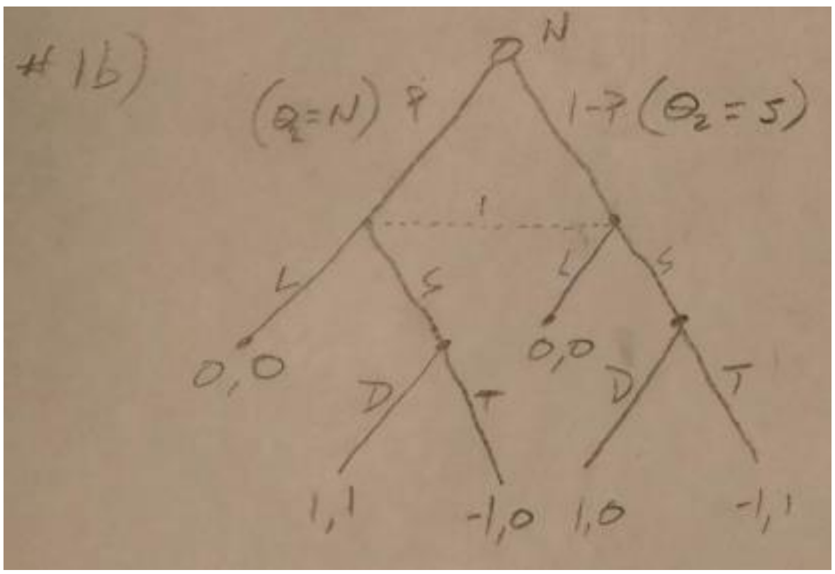
Matthew Draper

12/12/18

1a) See Figure 1. Equilibrium analysis reveals that Player 1 ought to trust Player 2 to deliver the problem set when $U_2(D, s_{-i}) > U_2(T, s_{-i})$, which will be true as long as $x > y$.



b) See Figure 2.



c) Assuming p = ¾, we get the table below:

1\2	$D_N D_J$	$D_N T_J$	$T_N D_J$	$T_N T_J$
L	0,0	0,0	0,0	0,0
S (Player 1)	3/4(1)+1/4(1)	3/4(1)+1/4(-1)	3/4(-1)+1/4(1)	3/4(-1)+1/4(-1)
S (Player 2)	3/4(1)+1/4(0)	3/4(1)+1/4(1)	3/4(0)+1/4(0)	3/4(0)+1/4(1)

Simplifying, we get:

1\2	$D_N D_J$	$D_N T_J$	$T_N D_J$	$T_N T_J$
L	0,0*	0,0*	0*,0*	0*,0*
S	1*, 3/4	1/2*, 1*	-1/2, 0	-1, 1/4

We have pure-strategy Bayesian Nash equilibria at {L, $T_N D_J$}, at {L, $T_N T_J$}, and at {S, $D_N T_J$}. We know these are BNE because both players' type-contingent strategies are best responses given their beliefs about other players' types (p = ¾) and their type-contingent strategies.[1]

To determine which of these equilibria (if any) are perfect Bayesian equilibria, we will consider each in turn. First, consider {L, $T_N D_J$}. PBE requires that each player's strategy be sequentially rational given each player's beliefs.[2] Our definition of sequential rationality requires that a strategy be a best response at each information set.[3] Consulting Figure 2, we observe that Player 2 is not playing a best response at either of her nodes, because she has a profitable deviation to a higher payoff in both cases. As such, {L, $T_N D_J$} cannot be a PBE.

Similarly, in the case of {L, $T_N T_J$}, Player 2 has a profitable deviation to D at her Θ_1=N information set, ruling it out as PBE. Finally, in the case of {S, $D_N T_J$} we observe that Player 2's strategy is sequentially rational when p = ¾. As such, we can characterize ({S, $D_N T_J$} p = 3/4) as a perfect Bayesian equilibrium. This equilibrium satisfies requirements #1-4, as it has a well-defined system of beliefs consistent with Bayes' rule (on the path of play), and both players are playing sequentially rational strategies.

d) We know that at p = ¾, {S, $D_N T_J$} is a PBE. After consulting Figure 2, we conclude that $D_N T_J$ is Player 2's only sequentially rational strategy that precludes any possibility of deviation. As a result, there can't be any value of p that establishes {L, $T_N D_J$} and {L, $T_N T_J$} as PBE.

---

[1] N.b. – Despite the fact that Player 2 knows her own type, her best response will track her probability-weighted type-dependent strategy (Wiens 05, Remark 6.0.1 pp. 13-14).
[2] Wiens 06, Requirement #4 (p. 9).
[3] Wiens 06, Definition 3.1 (p. 5).

2a) We will begin by characterizing both players' best-response functions. They are symmetrical. We'll adopt the convention of expressing the decision to protest as the value $p = 1$ and staying home as the value $p = 0$. The best responses are:

$$U_i(p_i, p_j; \theta_i) = \begin{cases} \theta^2 - 1/3 & \text{if } p_i = 1, p_j = 1 \\ \theta^2 - 2/3 & \text{if } p_i = 1, p_j = 0 \\ 0 & \text{if } p_i = 0, p_j = 0 \end{cases}$$

We want to identify the conditions under which each player will protest. This will occur when the expected value of protesting is higher than the expected value of staying home. The inequality we wish to solve (for Player 1 – symmetrical for Player 2) is:

$$u_1(p_1 = 1, p_2; \theta_1) \geq u_1(p_1 = 0, p_2; \theta_1)$$

In either case, each player's payoff will depend on the other player's behavior. Expected utility for either action is a function of that player's type and the other player's strategy. This gives us the following:

$$\theta_1^2 - 2/3 \geq \Pr(p_2 = 1 \mid \theta_1)\ (\theta^2 - 1/3) + (1 - \Pr(p_2 = 1 \mid \theta_1)\ (\theta^2 - 2/3)$$

Since $\Pr(p_2 = 1 \mid \theta_1) = \Pr(p_2 = 1)$, we have:

$$\theta_1^2 - 2/3 \geq \Pr(p_2 = 1)\ (\theta^2 - 1/3) + (1 - \Pr(p_2 = 1)\ (\theta^2 - 2/3)$$

Simplifying, we get:

$$2/3 \leq \theta^2 - 1/3\ ((1 - \Pr(p_2 = 1)\ (1/\theta^2) + (1 - \Pr(p_2 = 1)\ (1/\theta^2 - 2)$$

$$\theta_1^2 - 1/3 \geq 2/3 - ((1 - \Pr(p_2 = 1)\ (1/\theta^2) + (1 - \Pr(p_2 = 1)\ (1/\theta^2 - 2)$$

$$\theta_1^2 \geq 1 - ((1 - \Pr(p_2 = 1)\ (1/\theta^2) + (1 - \Pr(p_2 = 1)\ (1/\theta^2 - 2)$$

$$\theta_1 \geq \sqrt{}(1 - ((1 - \Pr(p_2 = 1) + (1 - \Pr(p_2 = 1)\ (1/\theta^2 - 2))$$

This equation gives us Player 1's (symmetrical) threshold type. They will protest only when the above inequality is true.[4] The players' threshold types are:

$$\hat{\theta}_1 \geq \sqrt{}(1 - ((1 - \Pr(p_2 = 1)\ (1/\theta^2) + (1 - \Pr(p_2 = 1)\ (1/\theta^2 - 2))$$

$$\hat{\theta}_2 \geq \sqrt{}(1 - ((1 - \Pr(p_1 = 1)\ (1/\theta^2) + (1 - \Pr(p_1 = 1)\ (1/\theta^2 - 2))$$

b) Since the players' strategies and payoffs are symmetrical and types are drawn from the uniform distribution, we observe that $\hat{\theta}_1 = \hat{\theta}_2$, which gives us:

$$BR_1 = \begin{cases} \{P\} & \text{if } \hat{\theta}_1 > \sqrt{}[1 - ((1 - \Pr(p_2 = 1)\ (1/\theta^2) + (1 - \Pr(p_2 = 1)\ (1/\theta^2 - 2)] \\ \{H\} & \text{if } \hat{\theta}_1 < \sqrt{}[1 - ((1 - \Pr(p_2 = 1)\ (1/\theta^2) + (1 - \Pr(p_2 = 1)\ (1/\theta^2 - 2)] \\ \{P,H\} & \text{if } \hat{\theta}_1 = \sqrt{}[1 - ((1 - \Pr(p_2 = 1)\ (1/\theta^2) + (1 - \Pr(p_2 = 1)\ (1/\theta^2 - 2)] \end{cases}$$

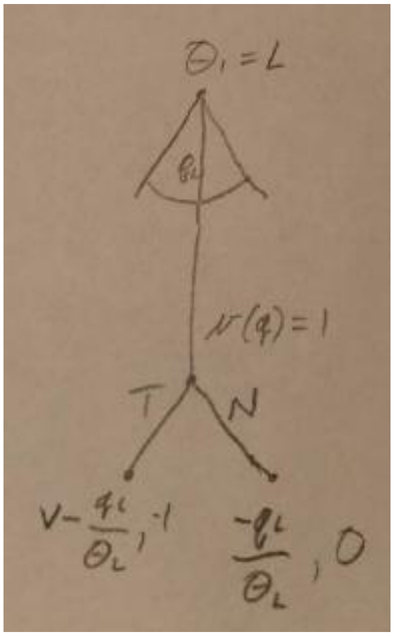
---

[4] N.b. – I know the inequality can be simplified further, but I'm reluctant to risk introducing more algebraic mistakes.

And similarly:

$$BR_2 = \begin{cases} \{P\} & \text{if } \hat{\theta}_2 > \sqrt{}[1 - ((1 - Pr(p_1 = 1)\ (1/\theta^2) + (1 - Pr(p_1 = 1)\ (1/\theta^2 - 2)] \\ \{H\} & \text{if } \hat{\theta}_2 < \sqrt{}[1 - ((1 - Pr(p_1 = 1)\ (1/\theta^2) + (1 - Pr(p_1 = 1)\ (1/\theta^2 - 2)] \\ \{P,H\} & \text{if } \hat{\theta}_2 = \sqrt{}[1 - ((1 - Pr(p1 = 1)\ (1/\theta^2) + (1 - Pr(p_1 = 1)\ (1/\theta^2 - 2)] \end{cases}$$

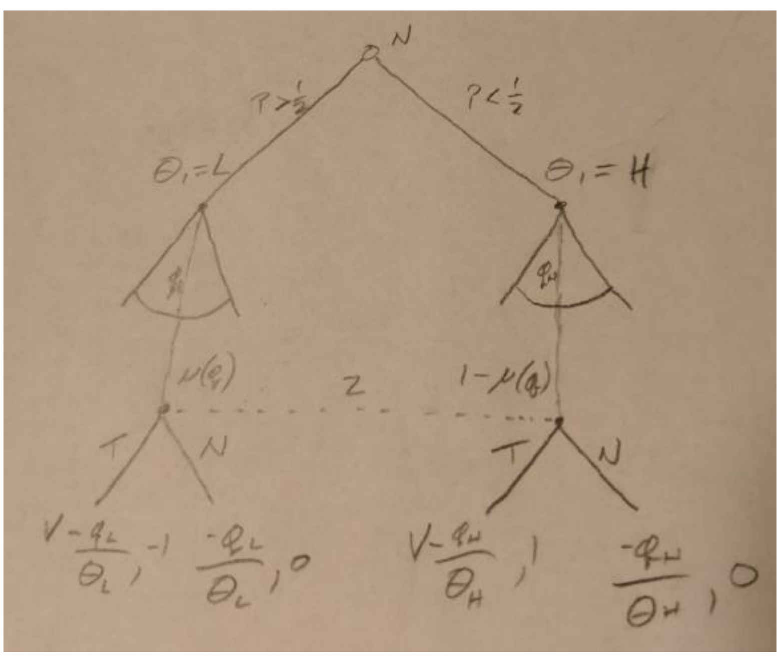
3a) See Figure 3.



b) In the simplified game, $\mu(q) = 1$. There can't be a subgame perfect equilibrium in which Player 2 plays T because it's impossible to specify a Nash equilibrium in each proper subgame. Since they know their decision node ($\mu(q) = 1$), Player 2 faces certain payoffs of -1 for granting tenure and 0 for not granting tenure. Since $0 > -1$, we can conclude that they will choose not to grant tenure.

Knowing this, Player 1 will evaluate their potential payoff of $-q_L/\theta_L$ ($\theta_L > 0$), and she will conclude that her payoff gets lower the more papers she publishes. This payoff will be "maximized" by producing 0 papers, for a net payoff of 0.

c) See Figure 4.



d) We will now look for pooling equilibria. Given the nature of pooling equilibria, we will assume that $q_H = q_L = q^*$, that is, the number of papers produced by Player 1 will be the same regardless of type. This means that the number of papers published will be uninformative for Player 2's choice, and as a result they won't update their beliefs. All they will know is that $\Pr(\theta_1 = H) < ½$ and that Player 1 has produced q* papers. $\theta_1 = H$

Given all this, Player 2 will only choose to play T if $U_2 (T, q^*) \geq U_2 (N, q^*)$. This gives us the following:

$p(1) + (1-p)(-1) > p(0) + (1-p)(0)$

$2p – 1 > 0$

$p > ½$

Since we know that $p < ½$, there cannot be any pooling equilibria where Player 2 plays T. Player 1 will thus not receive tenure. As we saw above, if Player 1 knows that Player 2 is going to play D, her incentive to produce papers evaporates and $q^* = 0$. This is the unique outcome of the pooling PBE. We must now define Player 2's off-the-path beliefs to support this pooling equilibrium. When $p > 0$, we can set these beliefs such that Player 2 believes that $\Pr(\theta_1 = L) = 1$. That will ensure that neither type of Player 1 has an incentive to deviate from $q^* = 0$ (for the reasons we saw in part b).

e) We will now look for separating equilibria. In a separating equilibrium, $q_H \neq q_L$, which means the number of papers produced by Player 1 will differ by type. This means that the number of papers produced by Player 1 is informative to Player 2, and Player 2 will update their beliefs accordingly, assigning a higher value of p if they observe q_H and a lower value if they observe q_L.

It would be irrational for either type of Player 1 to produce enough papers that she incurs a cost. If Player 1 were $\theta_1 = L$, she will incur a cost when $q_L > V\theta_L$, which is the value of tenure multiplied by her type's ability. Similarly for $\theta_1 = H$, costs will be incurred when $q_H > V\theta_H$. This establishes upper and lower bounds for q, and captures the intuitive result that high-competence types will be able to produce more papers than low competence types while still capturing some of the benefits of tenure. So:

$\mu_2(q) =$	$(\theta_1 = L)$	if $q = 0$
	$(\theta_1 = H)$	if $q > 0$

Since we know that $\theta_H > \theta_L$, it follows that $q_H > q_L$, which means that $\theta_1 = H$ will be able to produce more papers than $\theta_1 = L$. Player 1's $\theta_1 = L$ will therefore have no incentive to produce any papers ($q_L= 0$), and $\theta_1 = H$ will produce at least $q_H > 0$ papers. The updating of beliefs initiated by this information will induce Player 2 to grant tenure only when $q = q_H$ ($q_H > 0$). As long as $q_L > V\theta_L$, $\theta_1 = L$ will have no incentive to deviate. Similarly, as long as $q_H < V\theta_H$, $\theta_1 = H$ will have no incentive to deviate. This establishes a set of PBE, where $q_L= 0$ and $V\theta_H < q_H > 0$. It seems as though there would be an infinite number of these.

Player 2's off-the-path beliefs are unconstrained, so we can set them to make Player 1's on-path actions sequentially rational. We therefore set $\mu_2(q) = 1$ whenever $q > 0$. We know that on the path of play, when $q = 0$, Player 2 will choose N, and when $q > 0$, Player 2 will choose T. To sustain our candidate equilibrium, Player 2 will play {NN} when $q = 0$ and {TT} otherwise.

f) We answered this question while pursuing part e) above. As we saw, if Player 1 were $\theta_1 = L$, she will incur a cost when $q_L > V\theta_L$, which is the value of tenure multiplied by her type's ability. Similarly for $\theta_1 = H$, costs will be incurred when $q_H > V\theta_H$. To calculate the tenure threshold, all we need to do is set $\theta_1 = H$'s production of papers equal to $\theta_1 = L$'s upper bound, or in other words to have the high competence type produce the number of papers at which it's not rational for the low competence type to produce any more. As such, the minimum number of papers required to get tenure will be $q_H = V\theta_L$ (technically this would make Player 1 indifferent between granting and withholding tenure, so we might need to say $q_H > V\theta_L$).